OTKA REPORT

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1. Markov-Bernstein type inequalities

Let $L_p(K), 1 \leq p \leq \infty$ be the space of real functions on the set $K \subset \mathbb{R}^d$ endowed with the usual L_p norm

$$||f||_p := \left(\int_K |f|^p d\mu(\mathbf{x}) \right)^{1/p}.$$

Given a subspace $U \subset L_p(K)$ of differentiable functions the Markov problem consists in determining the norm of the differentiation operator on U, that is finding the quantity

$$M_p(U, K) := \sup_{u \in U} \frac{||Du||_p}{||u||_p}, \tag{1}$$

where $Du := |\partial u|$ and ∂u stands for the gradient vector of u. The study of the Markov problem in different function spaces has a long and fascinating history, Markov type inequalities are widely applied in various areas of analysis. In some instances the exact value of $M(U, K)$ can been found together with the corresponding extremal polynomials, in other cases only the asymptotic dependence on the dimension of U can be given. In particular it is well known that when $p = \infty$, $K \subset \mathbb{R}^d$ is a convex body and $M = P_n$ is the space of polynomials of degree at most n then $M(P_n, K)$ is of order n^2.

A. Markov type inequalities for multivariate polynomials in L_p norm, $1 < p < \infty$. [1] We obtained new Markov type results in case when $1 < p < \infty$. Namely, when $K \subset \mathbb{R}^d$ is a convex body we proved that

$$M_p(P_n, K) \leq c_d \frac{n^2}{r_K}$$

where r_K is the radius of the largest ball imbedded into K and c_d is some positive constant depending only on the dimension. If $K \subset \mathbb{R}^d$ is a star like domain with $C^\alpha, 0 < \alpha \leq 1$ boundary then we showed that $M_p(P_n, K) = O(n^{1+2/\alpha})$.

B. Exact Markov type inequalities for multivariate polynomials in L_2 norm. [3] In the case of L_2 norm our goal was to obtain exact value of the quantity $M_2(P_n, K)$ for certain domains. This goal was achieved for the Hermite weights when $K = \mathbb{R}^d$ and Hermite-Laguerre mixed weights for hyper quadrants in R^d. In particular, we proved that when

$$K = K_r = \{\mathbf{x} \in \mathbb{R}^d : x_k \geq 0, 1 \leq k \leq r\}, 1 \leq r \leq d, \;\; d\mu = \exp\left(- \sum_{1\leq k\leq r} x_k - \sum_{r+1\leq k\leq d} x_k^2 \right) dx$$

then $M_2(P_n, K_r) = \frac{1}{2\sin\frac{\pi}{4n+2}}$ with the extremal polynomial being a certain linear combination of Turan polynomials.

C. Exact Markov-Bernstein type inequalities for univariate k monotone polynomials. [2] We also found exact value of $M_p(U, K)$ for the case $d = 1, p = 1$ or ∞ and $U := P_n^k$ being the space of k monotone polynomials of degree n (polynomials with k nonnegative derivatives on $K = [-1, 1]$). For instance, it is shown that $M_\infty(P_n^k, K) = \frac{k-1}{1-x_1}$, where x_1 is the smallest root of a certain Jacobi polynomial.

2. Stability of the metric projection operator

Let X be a normed linear space and consider a subspace M of X. For a given $f \in X$ denote by $P_M f$ the set of best approximations to f from M, i.e.,

$$P_M f := \{m^* : m^* \in M, \|f - m^*\| = \inf_{m\in M} \|f - m\|\}.$$

P_M is said to be the **metric projection** operator onto M. We shall assume that $P_M f$ is non-empty, and also that $P_M f$ is single-valued, i.e., we have the uniqueness of the best approximation. A natural question that arises with respect to P_M is its **stability**. The stability of $P_M f$ with respect to *small perturbations of* f, that is continuity properties of the metric projection, have been widely investigated in the literature. In a joint paper with Allan Pinkus ([4]) we studied a new problem: How stable is the metric projection P_M relative to small perturbations of the closed linear subspace M? In order to address this question we need a measure of distance between subspaces:

$$d(M, N) := \max\left\{ \sup_{m\in M, \|m\|=1} \inf_{n\in N} \|m - n\|, \sup_{n\in N, \|n\|=1} \inf_{m\in M} \|n - m\| \right\}.$$

This measure (introduced by Krein, Krasnolselski in the Hilbert space setting) is symmetric in M and N, equals 0 if and only if $M = N$, and is a number between 0 and 1. When $X = H$ is Hilbert space then a proof of the fact that $\|P_M - P_N\| = d(M, N)$ was given by Akhiezer and Glazman. Our goal was to estimate $\|P_M f - P_N f\|$ in terms of $d(M, N)$. Typically such an estimate is of order $d(M, N)^\beta$ with some $\beta \leq 1$ which, in general, depends on the geometry of the space X. One approach to this problem is based on the *theory of strong uniqueness.* Strong uniqueness of best approximations has been extensively investigated over the past 40 years, see the recent survey Kroó, Pinkus [5]: Strong Uniqueness, Surveys in Approximation Theory , 2010. The metric projection P_M is said to be strongly unique of order $\alpha > 0$ at M if for each $f \in X$ and every $m \in M$ we have

$$\gamma_M(f)\|P_M f - m\|^\alpha \leq \|f - m\|^\alpha - \|f - P_M f\|^\alpha \qquad (*)$$

with some constant $\gamma_M(f) > 0$ depending only on f and M.

It turns out that we can estimate the stability of the metric projection under the assumption of strong uniqueness. Let M be a closed linear subspace of the normed linear space X such that the metric projection P_M satisfies the strong uniqueness condition (*). Then for any $f \in X$ and any closed linear subspace $N \subset X$

$$\|P_M f - P_N f\| \leq \frac{10}{\gamma_M(f)^{1/\alpha}} d(M,N)^{1/\alpha}.$$

Thus strong uniqueness of order α implies stability of order $\frac{1}{\alpha}$.

This yields sharp **order one** stability of metric projection in uniform norm.

Let M and N be closed linear subspaces of L^p, $1 < p < \infty$. Then from strong uniqueness results we obtain stability of orders $1/p$ and $1/2$, when $2 < p < \infty$ and $1 < p < 2$, respectively. On the other hand its is known that in a Hilbert Space we have $\|P_M - P_N\| = d(M,N)$, that is stability is of order 1 holds.

With a different approach under some additional conditions a sharper stability results for the metric projection in L^p, $p > 2$, can be exhibited. Let μ be a positive measure on a set K and $L^p(K,\mu)$ is the standard L^p-space. For a subspace M of $L^p(K,\mu)$ we say that M satisfies the Z_μ property if $\mu\{x : m(x) = 0\} = 0$ for every $m \in M$, $m \neq 0$.

Theorem. ([4]) *Consider the space $L^p(K,\mu)$, $p > 2$, where μ is a non-atomic measure. Let M be an r-dimensional subspace of $L^p(K,\mu)$ satisfying the Z_μ property. Then for every $f \in L^p(K,\mu)$, $p > 2$, there exists a constant $c_{M,f}$ depending on M and f such that for any r-dimensional subspace N of $L^p(K,\mu)$, $p > 2$, we have $\|P_M f - P_N f\|_p \leq c_{M,f} d(M,N)$, i.e., order 1 stability holds. Furthermore, the Z_μ property of M is necessary, in general for the above to hold.*

3. Approximation by homogeneous and convex multivariate polynomials

A. Weierstrass type theorems on convex bodies. Let us consider the problem of density of **homogeneous** polynomials on **0**-symmetric star like domains $K \subset \mathbb{R}^d$. Thus consider

$$H_n^d := \{\sum_{|\mathbf{k}|=n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R}\}, \quad H^d := \cup H_n^d.$$

Since uniform limits of bounded homogeneous polynomials must vanish in the interior of K the approximation problem becomes reduced to the boundary of K, i.e., the density problem for homogeneous polynomials should be considered only on ∂K. Since in general f is neither even nor odd, clearly at least **two homogeneous polynomials are need** for approximating f. The following conjecture was proposed by A. Kroó about 10 years ago:

*For any **0**-symmetric convex body $K \subset \mathbb{R}^d$ the set $H^d + H^d$ **is dense** in $C(\partial K)$, that is any function continuous on the boundary of K can be uniformly approximated there by a sum of 2 homogeneous polynomials.*

The density of $H^d + H^d$ in $C(\partial K)$ has been verified in 2 cases:

(i) When **d=2** and K is any **0**-symmetric convex body in $\mathbb{R}^2$(Benko-Kroó, Varju)

(ii) For any **0**-symmetric convex **polytope** in $\mathbb{R}^d, d > 2$ (Varju)

In a recent joint paper [6] A. Kroó and J. Szabados verified that the above conjecture holds for every **regular 0**-symmetric convex body $K \subset \mathbb{R}^d$, that is in case when K possesses a unique supporting hyper plane at every point of its boundary.

Theorem ([6]). *If the **0**-symmetric convex body $K \subset \mathbb{R}^d$ possesses a unique supporting hyper plane at every point of its boundary then $H^d + H^d$ is dense in $C(\partial K)$.*

B. Approximation of convex bodies by level surfaces of convex polynomials. Given a compact set $K \subset \mathbb{R}^d$ we consider the problem of approximating the boundary of K by level surfaces of algebraic polynomials defined by $L(p) := \{\mathbf{x} \in \mathbb{R}^d : p(\mathbf{x}) = 1\}, p \in P_n^d$. The measure of approximation is **Hausdorff distance** between sets denoted by $\rho(A, B)$. This problem is meaningful only when some restrictions are imposed on the approximating polynomials, otherwise Weierstrass theorem can be immediately applied.

When K is a convex body a natural restriction is convexity of approximating polynomials. Denote by CP_n^d the set of all polynomials of degree at most n which are **convex** on $\mathbb{R}^d$.

Problem. For a convex body $K \subset \mathbb{R}^d$ estimate the quantity

$$\Delta_n^c(K) := \inf\{\rho(\partial K, L(p)) : p \in CP_n^d\}.$$

This problem goes back to Minkowski who showed that any convex body can be approximated arbitrarily well by level surfaces of **convex analytic functions**. (Thus analytic convex functions are used instead of CP_n^d above.) Hammer verified that for any convex body $K \subset \mathbb{R}^d$: $\Delta_n^c(K) \to 0, n \to \infty$, that is any convex body can be approximated by convex algebraic level surfaces.

What can be said about the rate of $\Delta_n^c(K)$? Consider the Minkowski functional defined by $|\mathbf{x}|_K := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}$ and let us introduce its modulus of smoothness

$$\delta_K(t) := \sup_{\mathbf{x} \in \partial K, |\mathbf{h}|_{l_2} \leq 1} \{|\mathbf{x} + \mathbf{h}|_K + |\mathbf{x} - \mathbf{h}|_K - 2\}.$$

This modulus characterizes the smoothness of the boundary of the convex body K. It is easy to show that $\delta_K(t) = O(t)$ for any convex body. In addition, if K is regular (that is it possesses a unique supporting hyperplane at every point on its boundary) then $|\mathbf{x}|_K$ is differentiable and thus $\delta_K(t) = o(t)$. We obtained the next universal upper bound for the rate of approximation of convex bodies by convex algebraic level surfaces.

Theorem. (A.Kroó [7]) *For any convex body $K \subset \mathbb{R}^d$ of diameter ≤ 1*

$$\Delta_n^c(K) \leq cd^8 \delta_K\left(\frac{1}{n}\right), \quad n \in \mathbb{N},$$

with an absolute constant $c > 0$.

This implies several important corollaries.

A. For any convex body $K \subset \mathbb{R}^d$

$$\Delta_n^c(K) \leq \frac{cd^6}{n}.$$

B. For any **regular** convex body $K \subset \mathbb{R}^d$

$$\Delta_n^c(K) = o\left(\frac{1}{n}\right).$$

C. For any C^2 convex body $K \subset \mathbb{R}^d$ ($|\mathbf{x}|_K$ is twice continuously differentiable)

$$\Delta_n^c(K) \leq \frac{c}{n^2}.$$

4. Optimal polynomial meshes

Consider the space P_n^d of real algebraic polynomials of d variables and degree at most n. Let $K \subset \mathbb{R}^d$ be any compact set. Denote by $||p||_K := \sup_{\mathbf{x} \in K} |p(\mathbf{x})|$ the usual supremum norm on K. Moreover $card(Y)$ stands for the cardinality of a finite set Y.

A family of sets $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ is called an admissible mesh in K if there exist constants c_1, c_2 depending only on K such that

$$||p||_K \leq c_1 ||p||_{Y_n}, \quad p \in P_n^d, n \in \mathbb{N}$$

where the cardinality of Y_n grows at most polynomially, i.e. $card(Y_n) \leq c_2 n^m, n \in \mathbb{N}$ with some fixed $m \in \mathbb{N}$ depending only on K.

This notion of admissible meshes is related to **norming sets** widely used in the literature for the study of scattered data interpolation and cubature formulas on spheres. They are also applied for least squares approximation and construction of discrete extremal sets of Fekete and Leja type.

Since $\dim P_n^d \sim n^d$ we clearly must have $m \geq d$ in the above definition, provided that no polynomial vanishes on K. Of course, in optimal case we aim for a mesh with asymptotically minimal number of points in it, that is we would like to have $m = d$. This leads to the following definition.

Definition. *We shall say that an admissible mesh* $\boldsymbol{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ *in* $K \subset \mathbb{R}^d$ *is* **optimal** *if* $card(Y_n) \leq cn^d, n \in \mathbb{N}$ *with some* $c > 0$ *depending only on* K.

The basic question in this respect consists in describing those sets $K \subset \mathbb{R}^d$ which possess optimal or near optimal admissible meshes, in the sense that the cardinality of sets Y_n in the mesh $\mathbf{Y}$ does not grow too fast.

In the paper [8] we gave a systematic study of this question by considering two different categories of domains:

A) sets with certain analytic properties, i.e., graph domains bounded by graphs of polynomial, differentiable or analytic functions;

B) sets satisfying certain geometric properties, that is convex bodies, polytopes or star like domains.

In particular, it was shown that graph domains bounded by polynomial graphs, convex polytopes and star like sets with C^2 boundary possess **optimal admissible meshes**. In addition, it was verified that graph domains in $\mathbb{R}^d$ with piecewise analytic boundary and convex sets in $\mathbb{R}^2$ possess *almost* optimal admissible meshes in the sense that the cardinality of admissible meshes differs from the optimal only by a $\log n$ factor.

5. Christoffel Functions and universality limits for multivariate orthogonal polynomials

Let p_n denote the orthonormal polynomials of degree n with respect to a positive measure μ. Then the n-th reproducing kernel and Christofel function are defined by

$$K_n(x,t) := \sum_{j=0}^{n-1} p_j(x) p_j(t), \quad \lambda_n(\mu, x) := \frac{1}{K_n(x,x)},$$

respectively. Asymptotics for Christoffel Functions plays a crucial role in the analysis of orthogonal polynomials and weighted approximation. One application of the asymptotics for Christoffel Functions is the universality limits arising in the analysis of random matrices. Asymptotics for Christoffel

Functions and universality limits have been widely investigated for univariate orthogonal polynomials. In the multivariate setting asymptotics for Christoffel Functions has been found only for very specific weights. In the recent papers [9] and [10] A. Kroó and D. Lubinsky gave a comprehensive study of the asymptotics for Christoffel Functions for multivariate orthogonal polynomials for the rather general class of *regular* weights. These results were applied to the universality type limits on the ball and simplex.

6. Construction of approximating linear operators

To approximate functions on the real line by entire functions interpolating at equidistant nodes, using Freud-type weights, we constructed two types of operators [11]. The advantages of the first operator are that it is an exponential-type operator, interpolating at finitely many nodes; it uses only finitely many function values, while it is an infinite sum, and it allows to approximate functions growing exponentially at infinity. The latter is a new phenomenon compared to approximation of bounded and uniformly continuous functions by sinc-type functions. (The need for approximation by such sinc-type functions has been explored in numerous works of F. Stenger.) The second operator is an entire function which is not of exponential type, interpolating at infinitely many equidistant points. It allows the weighted approximation of a wider class of functions than the first operator (namely it allows weights for which there is no density of polynomials) and the rate of convergence is faster than in the first case.

Bernstein polynomials are a useful tool of approximating functions. We extended the applicability of this operator to a certain class of locally continuous functions [12]. To do so, we considered the Pollaczek weight

$$w(x) := \exp\left(-\frac{1}{\sqrt{x(1-x)}}\right), \qquad 0 < x < 1,$$

which is rapidly decaying at the endpoints of the interval considered. In order to establish convergence theorems and error estimates, we needed to introduce corresponding moduli of smoothness and K-functionals. Because of the unusual nature of this weight, we had to overcome a number of technical difficulties, but the equivalence of the moduli and K-functionals is a benefit interesting in itself.

7. Elliptic functions

Let **C, R, Z** denote the set of complex numbers, real numbers and integers, respectively. Let $0 \neq \omega_1, \omega_2 \in \mathbf{C}$, $\omega_1/\omega_2 \in \mathbf{C} \setminus \mathbf{R}$, and define $\Omega := \{m\omega_1 + n\omega_2 : m, n \in \mathbf{Z}\}$, an infinite set of lattice points in $\mathbf{C}$. We say that $z_1 \equiv z_2 \pmod{\Omega}$ if and only if $z_1 - z_2 \in \Omega$. Further let $Q := \{u\omega_1 + v\omega_2 : 0 \leq u, v < 1\}$, a half-open parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$. Let $E(\Omega)$ be the set of *elliptic*, or *doubly periodic* functions with respect to Ω: i.e. $f \in E(\Omega)$ iff f is meromorphic in $\mathbf{C}$ and

$$f(z + \omega_j) \equiv f(z) \quad \text{for all} \quad z \in \mathbf{C} \quad \text{and} \quad j = 1, 2.$$

These functions are periodic in two directions, thus they can be considered as analogues of real trigonometric polynomials. The existence of a non-constant $f \in E(\Omega)$ is not trivial. The name *elliptic* comes from the fact that such functions were discovered as inverse functions of elliptic integrals. The above definition is due to Weierstrass, and it is more convenient and simpler than the definition given via the so-called theta-functions of Jacobi. We listed some basic facts on elliptic functions, reconstructed them from given poles and zeros, and considered Lagrange, Hermite and Hermite–Fejér type interpolation problems [13].

8. Weighted Kantorovich operator

Since the Bernstein polynomials are not defined for $f \in L^p([0,1]),\ 1 \le p < \infty$, the Kantorovich polynomials K_n, given by

$$K_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{I_k} f(u)\,du, \tag{1}$$

$$I_k := \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right], \qquad p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k},\ x \in [0,1],$$

were introduced and studied. The case of weighted approximation by K_n operator with the Jacobi weight

$$w(x) = w^{\alpha,\beta}(x) = x^\alpha(1-x)^\beta, \qquad \alpha, \beta > -1,$$

was also investigated by Z. Ditzian and V. Totik. Some equivalence results were given under the restrictions

$$-\frac{1}{p} < \alpha,\ \beta < 1 - \frac{1}{p} \tag{2}$$

on the weight parameters. Here the left side inequalities for the weight parameters are necessary. In order to eliminate the right hand side inequalities, we constructed a weighted generalization of the classic Kantorovich operator as

$$B_n^*(f;x) = \sum_{k=0}^{n} \frac{\int_{I_k}(wf)(t)\,dt}{\int_{I_k} w(t)\,dt} p_{n,k}(x), \qquad x \in [0,1], \tag{3}$$

where

$$-\frac{1}{p} < \alpha,\ \beta, \qquad 1 \le p \le \infty$$

and

$$f \in L_w^p := \begin{cases} \{f \mid wf \in L^p(0,1)\} & \text{if} \quad 1 \le p < \infty, \\ \{f \mid f \in C(0,1),\ \lim_{x(1-x)\to 0}(wf)(x) = 0\} & \text{if} \quad p = \infty. \end{cases}$$

We gave convergence, direct and converse approximation results involving the weighted modulus of smoothness of second order, as well as a Voronovskaya-type asymptotic relation [15]. The saturation problem was solved as well [16].

9. Revisiting a classic theorem of Erdős and Grünwald

In 1938, Erdős and Grünwald investigated the behavior of the function

$$M_n(x) := \max_{1\le k\le n} \ell_{k,n}(x), \qquad |x| \le 1$$

where $\ell_{k,n}(x)$ are the fundamental functions of Lagrange interpolation based on the Chebyshev nodes. Their result implies that

$$\lim_{n\to\infty} \max_{|x|\le 1} M_n(x) = \frac{4}{\pi}.$$

Once this value is found, it is natural to ask for the behavior of the minimum of this function. In this connection we proved [17] that

$$\lim_{n\to\infty} \min_{|x|\le 1} M_n(x) = \frac{2}{\pi}\cos 2 - \frac{\sqrt{3}}{2}\pi = 0.580\ldots.$$

10. Laguerre–Pollaczek weights on the semiaxis

We considered the weight $u(x) = x^\gamma e^{x^{-\alpha} - x^\beta}$ with $x \in (0, +\infty)$, $\alpha > 0$, $\beta > 1$ and $\gamma \ge 0$, and prove Remez-, Bernstein–Markoff-, Schur- and Nikolskii-type inequalities for algebraic polynomials [18]. The interest in these problems is that they merge the difficultiess arising in connection with Laguerre type and Pollaczek type weights.

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