Final scientific report

1 Graph colorings

1.1 Nonrepetitive colorings

A vertex coloring of a graph is *nonrepetitive on paths* if there is no path $v_1, v_2, \ldots, v_{2t}$ such that v_i and v_{t+i} receive the same color for all $i = 1, 2, \ldots, t$. In [1], we determine the maximum density of a graph, which admits a k-coloring that is nonrepetitive on paths. We prove that every graph has a subdivision that admits a 4-coloring that is nonrepetitive on paths. The best previous bound was 5, proved by Grytczuk. Later, Pezarski and Zmarz improved it to best possible, a nonrepetitive 3-coloring. In [1], we prove that every graph with treewidth k and maximum degree Δ has a $O(k\Delta)$-coloring that is nonrepetitive on paths. We also study colorings that are nonrepetitive on walks, and provide a plausible conjecture, which would imply that every graph with maximum degree Δ has an $f(\Delta)$-coloring, which is nonrepetitive on walks. We prove that every graph with treewidth k and maximum degree Δ has a $O(k\Delta^3)$-coloring, which is nonrepetitive on walks.

The most intriguing open question in this field asks whether there exists a constant C such that every planar graph admits a nonrepetitive coloring with at most C colors. In joint work with Czap, we studied the following relaxation. If G is a plane graph, then a facial nonrepetitive vertex coloring of G is a vertex coloring such that any facial path is nonrepetitive. Let $\pi_f(G)$ denote the minimum number of colors of a facial nonrepetitive vertex coloring of G. Jendroľ and Harant posed a conjecture that $\pi_f(G)$ can be bounded from above by a constant. In [2], we prove that $\pi_f(G) \leq 24$ for any plane graph G. The subtlety is to introduce some extra edges, when complications may occur. The best lower bound we could provide, is 5. Since our ideas work with local colorings, we can naturally extend the result to graphs embedded in surfaces. We also prove the $n \times n$ grid requires at most 4 colors for a facial nonrepetitive coloring.

1.2 Crossings and colorings

A graph is planar, if it can be drawn in the plane without edge crossings. It follows from the Euler formula, that any planar graph has a vertex of degree at most 5. Therefore, using a greedy coloring algorithm, we can color the vertices of a planar graph by 6 colors. It is still easy to prove that 5 colors suffice. What happens if we allow some edge crossings in the drawing? Can we still 5 color the vertices? The answer is yes, if we allow at most two crossings, as it was shown by Oporowski and Zhao. The minimum number of crossings in a planar drawing of a graph G is its crossing number, denoted as $\mathrm{CR}(G)$. Is there any connection between the crossings number of a graph G and its chromatic number? Both of these graph parameters are hard to compute. Still, Albertson formulated a conjecture that for any r-chromatic G, the crossing number of G is at least the crossing number of K_r, the complete graph on r vertices. This conjecture is a weakening of Hajós conjecture and it includes the 4 color theorem as a subcase. In [12], we show the Albertson-conjecture follows from some known results up to a factor of 4. It is also true for $r \leq 16$, despite the fact that we only know the crossing number of K_r if $r \leq 12$. This improved earlier results by Albertson, Cranston and Fox. We also prove if G is an n-vertex, r-critical graph with $n \geq 3.57r$, then $\text{CR}(G) \geq \text{CR}(K_r)$. En route to our main result, we characterize the n-vertex r-critical graphs with $n = r + 3$ or $r + 4$. Our efforts have been acknowledged by Wikipedia, see http://en.wikipedia.org/wiki/Albertson_conjecture.

1.3 List colorings

In 1943, Hadwiger made the following conjecture, which is widely considered to be one of the most important open problems in graph theory.

Hadwiger Conjecture. Every K_t-minor-free graph is $(t-1)$-colorable.

The Hadwiger Conjecture holds for $t \leq 6$ and is open for $t \geq 7$. In fact, the following more general conjecture is open.

Weak Hadwiger Conjecture. Every K_t-minor-free graph is ct-colorable for some constant $c \geq 1$.

It is natural to consider analogous conjectures for list colorings. First, consider the choosability of planar graphs. Erdős, Rubin and Taylor conjectured that some planar graph is not 4-choosable, and that every planar graph is 5-choosable. The first conjecture was verified by Voigt and the second by Thomassen. Incidentally, Borowiecki asked whether every K_t-minor-free graph is $(t-1)$-choosable, which is true for $t \leq 4$ but false for $t = 5$ by Voigt's example. The following natural conjecture arises.

List Hadwiger Conjecture. Every K_t-minor-free graph is t-choosable.

The List Hadwiger Conjecture holds for $t \leq 5$. Again the following more general conjecture is open.

Weak List Hadwiger Conjecture. Every K_t-minor-free graph is ct-choosable for some constant $c \geq 1$.

In [8], we disprove the List Hadwiger Conjecture for $t \geq 8$, and prove that $c \geq 4/3$ in the Weak List Hadwiger Conjecture: for every integer $t \geq 1$,
(a) there is a K_{3t+2}-minor-free graph, which is not $4t$-choosable.
(b) there is a K_{3t+1}-minor-free graph, which is not $(4t-2)$-choosable,
(c) there is a K_{3t}-minor-free graph, which is not $(4t-3)$-choosable.

2 Graph decompositions

A graph G has an *H-decomposition*, if the edges of G can be decomposed into subgraphs isomorphic to H. There is a necessary condition: $|E(H)|$ divides $|E(G)|$. In what follows, we always assume this hypothesis. The general problem of H-decompositions was proved to be NP-complete for any H of size greater than 2 by Dor and Tarsi. However, Barát and Thomassen posed the following

Conjecture For each tree T, there exists a natural number k_T such that the following holds: if G is a k_T-edge-connected graph such that $|E(T)|$ divides $|E(G)|$, then G has a T-decomposition.

In 2008, Thomassen proved that every 207-edge-connected graph G has a set E of at most 6 edges such that $G - E$ has a 4-path-decomposition. Slightly after that, Thomassen proved every 171-edge-connected graph of size divisible by 3, has a 3-path-decomposition. This latter was our starting point in [4]. Let Y be the unique tree with degree sequence $(1, 1, 1, 2, 3)$. We prove that if G is a 287-edge-connected graph of size divisible by 4, then G has a Y-decomposition. The proof consists of three main ingredients. In principle, the method could be applied to any tree T. Let G be a graph of sufficiently high edge-connectivity, and let T be a tree on k edges. In a nutshell:

1. Remove copies of T from G such that a bipartite graph $G[A, B]$ remains, which still contains many edge-disjoint spanning trees.
2. Remove more copies of T such that each degree in A becomes divisible by k, and the rest still contains some edge-disjoint spanning trees.
3. Group the edges from A such that copies of T arise, which altogether decompose the rest.

In [4], we show that step 1. is always possible. Therefore, it is sufficient to prove the Barát-Thomassen conjecture for bipartite graphs. During the period of the OTKA Grant, Thomassen proved one of our intended goals. He validated the above **Conjecture** for an arbitrary star. In particular for the 3-star, he proved that 8-connectivity suffices, thereby proving a weak version of Tutte's 3-flow conjecture.

2.1 A discrete geometry application

We build an $n \times n \times n$ cube from n^3 unit cubes. Every small cube has six faces. Two unit cubes are neighbors, if they touch along a face. We identify the n^3 unit cubes with the vertices of a graph, Q_n. Two vertices are *neighbors*, if the corresponding two unit cubes touch along a face. There is a characterization theorem by Hoffman, showing that Q_n has a 3-star decomposition. Our subject in [9] is to successively remove vertices of degree 3 from Q_n. This process is a *dismantling* of the cube. Clearly a dismantling gives a decomposition, but not vice versa. How many vertices of Q_n can we remove? A small geometric argument shows, that in the extremal case, there must be n^2 independent vertices of Q_n left. In [9], we show that for any integer n, $n \geq 2$, there is a dismantling process ending with n^2 independent vertices. In particular cases, the three orthogonal projections of the n^2 independent cubes form a complete square. These are perfect solutions, which correspond to Latin squares. As it turns out from our computer search, most solutions are imperfect. If n is a power of 2, we construct an imperfect solution, which has vertices incident to three orthogonal faces of Q_n. We conjecture, that this is a solution with the smallest possible area, when projected to three orthogonal faces. In [9], we pose numerous questions for further research.

2.2 Islands

In the 80's, Akiyama and Kano used factors and decompositions of graphs to study the existence of tilings of the grid.

Two players can play the following game on an $m \times n$ board. The first player occupies some disjoint 2×2 squares of the board. After that, the second player tries to fill up the remaining space with 2×2 squares and special dominoes. Subject to the uncovered area, the first player wants to maximize, the second player wants to minimize. In [6], we show the somewhat unusual phenomenon that there is an equilibrium at $\frac{nm}{5}$. The second player can always reach this, whatever the first player selected in his turn. On the other hand, the first player has a construction leaving $\frac{nm}{5}$ empty squares such that the second player can not make any move.

There is a strong connection between these tilings of the grid and the following theory of islands via a dualization.

Let a rectangular $m \times n$ board be given. We associate a number (real or integer) to each cell of the board. We can think of this number as a height above see level. A rectangular part of the board is a *rectangular island* if and only if there is a possible water level such that the rectangle is an island in the usual sense. It is a fundamental property that two islands are either containing or disjoint. The maximum number of rectangular, brick and triangular islands have been recently determined by Czédli; Pluhár; Horváth, Németh and Pluhár. In [5], we give a much shorter proof of these results, and also analogous results on the toroidal and some other boards.

3 Diverse

In [10], we collect 19 open questions and conjectures attributed to Carsten Thomassen. Most of them are unsolved, and we describe the history of the problems and the present state of art. Several of these questions are connected to graph colorings or decompositions. The paper has received much attention as shown by the on-line statistics engine Top25. It is listed among the Top25 Hottest Articles as No.9. in the most read mathematical papers in July-September 2010 category.

http://top25.sciencedirect.com/subject/mathematics/16/archive/28

In the journal Discrete Mathematics, it was placed 1, 3, 1, 11, among the most read papers in the last four periods.

3.1 Combinatorial games

In [11], we analyze the duration of the unbiased Avoider-Enforcer game for three basic positional games. All the games are played on the edges of the complete graph on n vertices, and Avoider's goal is to keep his graph outerplanar, diamond-free and k-degenerate, respectively. It is clear that all three games are Enforcer's wins, and our main interest lies in determining the largest number of moves Avoider can play before losing.

Extremal graph theory offers a general upper bound for the number of Avoider's moves. As it turns out, for all three games we manage to obtain a lower bound that is just an additive constant away from that upper bound. In particular, we show a strategy for Avoider to keep his graph outerplanar for at least $2n-8$ moves, being just 6 short of the maximum possible. A diamond-free graph can have at most $d(n) = \lceil \frac{3n-5}{2} \rceil$ edges, and we prove that Avoider can play for at least $d(n) - 3$ moves. Finally, if k is small compared to n, we show that Avoider can keep his graph k-degenerate for as many as $e(n)$ moves, where $e(n)$ is the maximum number of edges in a k-degenerate graph.

3.2 Extremal set systems

For a property Γ and a family of sets $\mathcal{F}$, let $f(\mathcal{F}, \Gamma)$ be the size of the largest subfamily of $\mathcal{F}$ having property Γ. For a positive integer m, let $f(m, \Gamma)$ be the minimum of $f(\mathcal{F}, \Gamma)$ over all families of size m. A family $\mathcal{F}$ is B_d-free if it has no subfamily $\mathcal{F}' = \{F_I : I \subseteq [d]\}$ of 2^d distinct sets such that for every $I, J \subseteq [d]$, both $F_I \cup F_J = F_{I \cup J}$ and $F_I \cap F_J = F_{I \cap J}$ hold. A family $\mathcal{F}$ is a-union-free if $F_1 \cup \ldots \cup F_a \neq F_{a+1}$, whenever $F_1, \ldots, F_{a+1}$ are distinct sets in $\mathcal{F}$. We verify a conjecture of Erdős and Shelah that $f(m, B_2\text{-free}) = \Theta(m^{2/3})$. We also obtain lower and upper bounds for $f(m, B_d\text{-free})$ and $f(m, a\text{-union-free})$. Our upper bound on $f(m, a\text{-union-free})$ turned out to have the correct order of magnitude, as later Fox, Lee and Sudakov proved a matching lower bound.

References

[1] Barát J, Wood DR: Notes on nonrepetitive graph coloring, Electron J Combin 15: R99, p.13, 2008

[2] Barát J, Czap J: Vertex coloring of plane graphs with nonrepetitive boundary paths, J Graph Theor: p.8., submitted

[3] Barát J, Füredi Z, Kantor I, Kim Y, Patkós B: Large B_d-free and union-free subfamilies, SIAM J Disc Math: p.8., to appear

[4] Barát J, Gebner D: Edge-decomposition of graphs into copies of a tree with four edges, J Graph Theory: p.12., submitted

[5] Barát J, Hajnal P, Horváth EK: Elementary proof techniques for the maximum number of islands., Eur J Combin 32: 276-281, 2011

[6] Barát J, Hajnal P: Saturated tilings with dominoes and 2x2 squares, Proceedings of the 6th Japanese-Hungarian Symposium on Discrete Mathematics. Budapest, Hungary, p.9., 2009

[7] Barát J, Hajnal P, Lin Y, Yang A: On the structure of graphs with pathwidth at most two, Stud Sci Math Hung: p.12., to appear

[8] Barát J, Joret G, Wood DR: Disproof of the List Hadwiger Conjecture, Electron J Comb: p.7., to appear

[9] Barát J, Korondi M, Varga V: How to dismantle an atomic cube in zero gravity, Proceedings of the 7th Hungarian-Japanese Symposium on Discrete Mathematics and its Applications, p.9., 2011

[10] Barát J, Kriesell M: What is on his mind?, Discrete Math 310: 2573-2583., 2010

[11] Barát J, Stojakovic M: On winning fast in Avoider-Enforcer games, Electron J Combin 17: R56 p.12., 2010

[12] Barát J, Tóth G: Towards the Albertson conjecture, Electron J Combin 17: R73, p.15., 2010