Final report for the project “OTKA FK-19 131895”

1. Domestic hot water production with solar heating systems

In the second year of the project, we published papers [1] and [2]. In the papers, we worked out a new analytical system solution for a rather basic and frequently used two-dimensional ordinary differential equation model of complete collector-storage systems. The solution provides easy- and fast-to-use explicit formulas to determine both the temperature of the collector and the temperature of the storage.

The analytical solution was validated with the measured data of the experimental solar heating system set up in the first project year. As a part of the validation of the (two-dimensional) system solution, the analytical solution of the (one-dimensional) solar collector sub-model (which is known in the literature) was also validated separately with the measured data of the collector itself. Based on the validation containing a relatively long time period of 19 days, the error of the new system solution is 3.4% in case of the collector and 2.0% in case of the storage, which means an outstanding precision in the field.

For a continuation of the above work, we worked out a new control method based on the above analytical solution. We also completed the existing measured solar heating system with controlling equipment. With the aim of it, we will be able to test in practice the expectable new control methods. The theoretical results are ready, their validation and the preparation of the corresponding manuscript, to be published in a high quality (Q1) journal, are in progress. For additional information, the half-prepared manuscript (titled (in Hungarian) “Arányos szabályozás és konstans térfogatáram...”) is attached to this report in the *Appendix* below. Although it is relatively raw, it essentially contains the new results in the form of mathematical formulas.

In the framework of the present project, we also developed an invention: It is a new solar device (that can be called solar pot) that serves for liquid heating and/or cooking, mostly for meals purposes, by means of the rather direct use of solar energy harvested by solar collectors. By now, the protection process has been successfully finished at the Hungarian Intellectual Property Office. (The cost of the patent process itself has been met by the financial support of an application that we gained inside our University (not by the present OTKA FK_19 fund).) Now our invention (utility model) is protected officially according to the following specification: patent number 5489, title (translated from Hungarian): Equipment for preparing food by means of solar irradiation and/or for heating liquid [3].

In the third year of the project, we worked out a black-box type multiple linear regression based mathematical model for (real-time) predicting the indoor temperature of mobile (office) containers. The model, having low computational demand, could be generalized for other types of residence places as well. We validated the model based on measured data, which has the following (exogenous) inputs: global solar irradiance, environment temperature and wind speed. Then we used the model to estimate the application potential of (hot water producing) solar collectors, installed on the top of containers, for space heating. Based on the results with respect to a single room container, with an inner area of about 5 m^2, two solar collectors (having about 4 m^2 area in sum) could extend the time with comfortable indoor temperature by more than 5 hours within a three-day period in spring (in Hungary), without any other auxiliary heating. These results have been published as a journal article in [4].

2. Exploitation of thermal water resources

Based on the hydrogeological expertise of our Italian partners, we developed a game-theoretical model for the sustainable use of thermal water resources in the case of the Ischia volcanic island (Italy). In the game the spas are the players, the strategy of a player consists of a fixed pumping rate and daily time durations of pumping. The payoff of a player is proportional to the total quantity of withdrawn thermal water in a given time period. For the sustainability, the total pumping should be limited, since overpumping implies a significant recharge of the aquifers from seawater, changing the quality of the thermal water. For the obtained *n*-player normal form game, a special constrained Pareto optimal strategy choice is obtained, considered as solution of the game. The results have been discussed in international conferences [5], [6], [7], and published in paper [8].

Going beyond the planned research, we started to collaborate with the Italian partner concerning the sustainability of two economic activities in the Tivoli Plain (Lazio Region, Italy): There are large number of active quarries in the area extracting travertine, needing massive dewatering, implying a substantial drawdown measured at the Acque Albule spring used by the Terme di Roma Spa, the other important economic actor in the area, and the water level of the near Aniene river is also an issue. This is a promising research line, since similar sustainability conflicts are known in Hungary, too. During the prolongation of the project, by now the Hungarian part of the involved research group already prepared the draft of the mathematical and computational part of a paper, and an abstract has been already submitted to the following international conference: "The Geoscience paradigm: Resources, Risk and future perspectives, 19-21 Sept. 2023, Potenza, Italy" [9]. Furthermore, a manuscript on the results is to be published in a high quality (Q1) journal. For additional information, the half-prepared manuscript (titled "Game-theoretical model for sustainable use of groundwater…") is attached to this report in the *Appendix* below. Although it is relatively raw, it essentially contains the new results mostly in the form of mathematical formulas. The hydrogeological part is in preparation by the Italian partner.

Conflicts on thermal water resources can often be described and solved mathematically with bimatrix games in case of two consumers. Pareto optimality is often used as the (cooperative) solution of such resource conflicts since they are not improvable in the following sense: In comparison with a Pareto optimal equilibrium (strategy pair) of the game, there are no such strategies for the consumers with which at least one of their payoffs (profits) would increase while the other's payoff would not decrease. We worked out a new theorem, along with an algorithm, which makes it easier to check the Pareto optimality of a strategy pair in bimatrix games and, therefore, to solve certain conflict situations among consumers of thermal water resources. Then we proposed a nonlinear transform to make the payoff functions of the consumers linear with respect to the new transformed independent variables of the game. This transform makes it even more convenient to check the Pareto optimality of strategy pairs in certain cases. These results have been published as a journal article in [10].

With respect to the mathematical description of resource allocation problems, like allocating/distributing (limited) thermal water resources among several users, we proved another new theorem (along with two new lemmas) for the explicit expression of properly assigned (dependent) variables by means of the other (independent) variables in a system of inequality and quadratic equality constraints. The sum of the (nonnegative) variables may be either prefixed (if the available resource is limited) or not. The constraints may describe the feasible set in various resource allocation tasks (possibly in optimization or game-theoretical contexts) or in other problems. Furthermore, a practical algorithm was derived for assigning in a feasible way the independent variables (corresponding to the freely adjustable portions of the used thermal water to be distributed), to which (possibly limited) arbitrary nonnegative values can be prescribed. Practical examples were provided to facilitate utilizing the results. This work has been published as a journal article in [11].

3. Further achievement in the framework of the project

We organized the 1st International Conference on Efficiency, Solar and Thermal Energy for the Human Comfort on 9 July 2021 in the Hungarian University of Agriculture and Life Sciences (former Szent István University), Gödöllő, Hungary. Together with the University, we issued the Book of Abstracts of the Conference [12]. The editors – Gábor Géczi, Richárd Kicsiny and László Székely – are all members of the current OTKA FK_19-supported research project, the data of which (National Research, Development and Innovation Office (Hungary), Grant No. 131895) we indicated in the Book of Abstracts. The Conference had 25 participants, among which 8 were foreigner researchers.

4. Publications

Our results were discussed in international conferences [2], [5], [6], [7], and published as journal articles [1], [4], [8], [10], [11]. Additionally, an abstract [9] is already submitted to an international conference and two further papers are under preparation. Furthermore, our invention [3] was protected at the Hungarian Intellectual Property Office and we organized an international conference with a book of abstracts [12].

Sorszám	Közleményjegyzék	Dokumentum típusa	Impakt faktor	NKFI támogatás feltüntetve?	Támogató szervezetek
[1]	Székely, L., Kicsiny, R., Hermanucz, P., Géczi, G. (2021): Explicit analytical solution of a differential equation model for solar heating systems. *Solar Energy*, 222, pp. 219-229.	paper in scientific journal	5.742 (Q1)	yes	
[2]	Székely, L., Kicsiny, R., Hermanucz, P., Géczi, G.: Explicit analytical solution of some differential equation models for solar heating systems. *International Conference on Efficiency, Solar and Thermal Energy for the Human Comfort* 9 July 2021, Gödöllő, Hungary. Book of Abstracts, pp. 23-24. (Editors: Gábor Géczi, Richárd Kicsiny and László Székely, ISBN 978-963-269-958-5)	conference lecture		yes	
[3]	*Equipment for preparing food by means of solar irradiation*, utility model, Hungarian Intellectual Property Office, patent number 5489, owner: Hungarian University of Agriculture and Life Sciences, inventors: Géczi, G., Kicsiny, R.	Hungarian utility model		no	Hungarian University of Agriculture and Life Sciences
[4]	Patonai, Z., Kicsiny, R., Géczi, G. (2022): Multiple linear regression based model for the indoor temperature of mobile containers. *Heliyon*, 8 (12), e12098	paper in scientific journal	3.776 (Q1)	yes	

[5]	Piscopo, V., Kicsiny, R., Scarelli, A., Varga, Z.: The thermal waters of the Isle of Ischia (Southern Italy): The hydrogeological support to define the sustainable yield. *International Conference on Efficiency, Solar and Thermal Energy for the Human Comfort* 9 July 2021, Gödöllő, Hungary. Book of Abstracts, pp. 11-12. (Editors: Gábor Géczi, Richárd Kicsiny and László Székely, ISBN 978-963-269-958-5)	conference lecture		no	Department of Ecological and Biological Sciences, Tuscia University (Italy).
[6]	Kicsiny, R., Piscopo, V., Scarelli, A., Varga, Z.: Game-theoretical model for the sustainable allocation of thermal water resources. A case study on the Ischia Island. *International Conference on Efficiency, Solar and Thermal Energy for the Human Comfort* 9 July 2021, Gödöllő, Hungary. Book of Abstracts, pp. 13-14. (Editors: Gábor Géczi, Richárd Kicsiny and László Székely, ISBN 978-963-269-958-5)	conference lecture		yes	Department of Ecological and Biological Sciences, Tuscia University (Italy).
[7]	Kicsiny, R., Piscopo, V., Scarelli, A., Varga, Z.: Game-theoretical model for the sustainable allocation of thermal water resources. A case study on the Ischia island. *In*: Alfonso Corniello, Emilio Cuoco, Paolo Fabbri, Giovanni Forte, Vittorio Paolucci, Vincenzo Piscopo, Dario Tedesco, Stefano Viaroli (Eds), *3rd International Multidisciplinary Conference on Mineral and Thermal Waters*, Caserta (Italy), June 26–30, 2022, Conference Abstract Book, IAH Italy - Associazione Internazionale degli Idrogeologi Sezione Italiana, p. 73.	conference lecture		no (for technical reason), but yes in the full paper version [8]	Department of Ecological and Biological Sciences, Tuscia University (Italy).
[8]	Kicsiny, R., Piscopo, V., Scarelli, A. and Varga, Z. (2022): Game-theoretical model for the sustainable use of thermal water resources: the case of Ischia volcanic island (Italy). *Environmental Geochemistry and Health*, 44 (7), pp. 2021-2035.	paper in scientific journal	4.898 (Q1)	yes	Department of Ecological and Biological Sciences, Tuscia University (Italy).
[9]	Piscopo V., Sebestyén Z., Sbarbati C. & Varga Z., Game-theoretical model for	conference lecture		no (for technic	

	sustainable use of groundwater in the heavily stressed system of the Acque Albule Basin (Rome, Italy). Abstract submitted to the conference "*The Geoscience paradigm: Resources, Risk and future perspectives, 19-21 Sept. 2023, Potenza, Italy*."			al reason), but will be yes in the paper version	
[10]	Kicsiny, R., Varga, Z. (2022): New algorithm for checking Pareto optimality in bimatrix games. *Annals of Operations Research*, 320, pp. 235-259.	paper in scientific journal	4.820 (Q1)	yes	
[11]	Kicsiny, R., Varga, Z., Hufnagel, L. (2022): Allocation of limited resources under quadratic constraints. *Annals of Operations Research*, 322, pp. 793-817.	paper in scientific journal	4.820 (Q1)	yes	
[12]	Géczi, G., Kicsiny, R., Székely, L. (editors): *Efficiency, solar and thermal energy for the human comfort*, Book of Abstracts, 2021, Hungarian University of Agriculture and Life Sciences, Gödöllő, Hungary, p. 48. (international conference), ISBN 978-963-269-958-5	conference book of abstracts		yes	

5. Level of completeness

With respect to the topic of Domestic hot water production with solar heating systems, we estimate that our level of completeness compared to the work amount planned originally in the project is more than 100%. With respect to the topic of Exploitation of thermal water resources, we estimate that our level of completeness compared to the work amount planned originally in the project is more than 95%. The total level of completeness compared to the originally planned work amount is more than 100%.

Appendix

For additional information, in this appendix, two half-prepared manuscripts (titled “Arányos szabályozás és konstans térfogatáram...” and “Game-theoretical model for sustainable use of groundwater…”) worked out in the framework of the present project are enclosed.

Arányos szabályozás és konstans térfogatáram, átlagos hőmérséklettel

GG, KR, SZL

May 3, 2023

1 Introduction

2 Models

2.1 Nomenclature

2.2 General model for hőcserélő nélküli rendszerre kevert tartály és arányos szabályozás esetén

Consider the differential equation

$$\begin{cases} \dot{T}_c = \dfrac{A_c\eta_0}{\rho_c c_c V_c} I_c + \dfrac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \dfrac{v}{V_c}(T_s - T_c), \\ \dot{T}_s = \dfrac{v}{V_s}(T_c - T_s), \end{cases} \tag{1}$$

where

$$T_{av} = \frac{T_c + T_{cold}}{2}. \tag{2}$$

Arányos szabályozás esetén

$$v = a(T_c - T_s), \tag{3}$$

then

$$\begin{cases} \dot{T}_c = -\dfrac{a}{V_c}(T_s - T_c)^2 + \dfrac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \dfrac{A_c\eta_0}{\rho_c c_c V_c} I_c, \\ \dot{T}_s = \dfrac{a}{V_s}(T_s - T_c)^2. \end{cases} \tag{4}$$

A kollektor hőveszteségét elhagyva

$$\begin{cases} \dot{T}_c = -\dfrac{a}{V_c}(T_s - T_c)^2 + \dfrac{A_c \eta_0}{\rho_c c_c V_c} I_c, \\ \dot{T}_s = \dfrac{a}{V_s}(T_s - T_c)^2. \end{cases} \tag{5}$$

By taking the difference of the two equations of system (5) and introducing the new variable

$$x = T_c - T_s \tag{6}$$

we get the following separable differential equation

$$\dot{x} = -ac_1 x^2 + c_2, \tag{7}$$

where

$$c_1 = \frac{1}{V_c} + \frac{1}{V_s}, \qquad c_2 = \frac{A_c \eta_0}{\rho_c c_c V_c} I_c. \tag{8}$$

Note that both c_1 and c_2 are positive. According to this the quadratic function

$$f(x) = -ac_1 x^2 + c_2 \tag{9}$$

on the right side of equation (7) has two distinct real roots of opposite sign

$$x_1 = -\sqrt{\frac{c_2}{ac_1}}, \qquad x_2 = \sqrt{\frac{c_2}{ac_1}}, \tag{10}$$

which also means that f can be factorized:

$$f(x) = -ac_1 x^2 + c_2 = -ac_1 \left(x - \sqrt{\frac{c_2}{ac_1}} \right) \left(x + \sqrt{\frac{c_2}{ac_1}} \right). \tag{11}$$

It follows as well that the differential equation (7) has two equilibria, namely x_1 and x_2. Note that from a practical point of view only the case $x(0) = T_c(0) - T_s(0) > 0$ case is relevant. Observe that x_2 is a locally asymptotically stable equilibrium, that is for all solutions with $x(0) > 0$

$$\lim_{t \to \infty} x(t) = x_2.$$

For all solution for which $x(0) \in]0, x_2[$ are strictly monotonically increasing, furthermore all solution for which $x(0) \in]x_2, \infty[$ are strictly monotonically increasing due to the sign of f on these intervals.

IDE TEGYÜNK EGY ÁBRÁT AZ 1D-S FÁZISDIAGRAMMRÓL?

Statement 1. *Depending on $x(0) > 0$ the solution of differential equation (7) is the following for all $t \in [0, \infty[$:*

(a) if $x(0) \in]0, x_2[$ then

$$x(t) = \sqrt{\frac{c_2}{ac_1}} \left(1 - \frac{2}{\frac{\sqrt{ac_1}x(0)+\sqrt{c_2}}{\sqrt{c_2}-\sqrt{ac_1}x(0)} e^{2\sqrt{c_1 c_2 a}t} + 1} \right), \tag{12}$$

(b) if $x(0) = x_2$ then

$$x(t) \equiv x_2, \tag{13}$$

(c) if $x(0) \in]x_2, \infty[$ then

$$x(t) = \sqrt{\frac{c_2}{ac_1}} \left(1 + \frac{2}{\frac{\sqrt{ac_1}x(0)+\sqrt{c_2}}{\sqrt{c_2}-\sqrt{ac_1}x(0)} e^{2\sqrt{c_1 c_2 a}t} + 1} \right). \tag{14}$$

Proof. Case (b) is obvious. We prove only Case (a), the proof of Case (c) is analogous to Case (a), the details of it are left to the Reader.

Since $x(0)$ is neither x_1 nor x_2 then the solution of separable differential equation (7) can be obtained in the following way. Rewriting equation (7) into the form

$$\frac{\dot{x}}{-ac_1 x^2 + c_2} = 1, \tag{15}$$

and integrating with respect to time over the interval $[0, t]$ leads to

$$\int_0^t \frac{1}{-ac_1 x^2(s) + c_2} \cdot \dot{x}(s) ds = \int_0^t 1 ds. \tag{16}$$

A substitution in the integral on the left side of the equation gives

$$\int_{x(0)}^{x(t)} \frac{1}{-ac_1 u^2 + c_2} du = t. \tag{17}$$

Factorization yields

$$-\frac{1}{ac_1} \int_{x(0)}^{x(t)} \frac{1}{\left(u - \sqrt{\frac{c_2}{ac_1}} \right) \left(u + \sqrt{\frac{c_2}{ac_1}} \right)} du = t. \tag{18}$$

One can easily check (e.g. using partial fraction decomposition) that the integral can be written in the form

$$-\frac{1}{ac_1}\int_{x(0)}^{x(t)} \frac{\frac{\sqrt{ac_1}}{2\sqrt{c_2}}}{u-\sqrt{\frac{c_2}{ac_1}}} - \frac{\frac{\sqrt{ac_1}}{2\sqrt{c_2}}}{u+\sqrt{\frac{c_2}{ac_1}}} du = t. \tag{19}$$

Using Newton–Leibniz Theorem and the assumption on $x(0)$ gives

$$\frac{1}{2\sqrt{ac_1c_2}}\left[\ln\left|u+\sqrt{\frac{c_2}{ac_1}}\right| - \ln\left|u-\sqrt{\frac{c_2}{ac_1}}\right|\right]_{x(0)}^{x(t)} = t, \tag{20}$$

$$\left[\ln\frac{\left|u+\sqrt{\frac{c_2}{ac_1}}\right|}{\left|u-\sqrt{\frac{c_2}{ac_1}}\right|}\right]_{x(0)}^{x(t)} = 2\sqrt{ac_1c_2}t, \tag{21}$$

$$\ln\frac{\left|x(t)+\sqrt{\frac{c_2}{ac_1}}\right|}{\left|x(t)-\sqrt{\frac{c_2}{ac_1}}\right|} - \ln\frac{\left|x(0)+\sqrt{\frac{c_2}{ac_1}}\right|}{\left|x(0)-\sqrt{\frac{c_2}{ac_1}}\right|} = 2\sqrt{ac_1c_2}t, \tag{22}$$

$$\ln\left(\frac{\left|x(t)+\sqrt{\frac{c_2}{ac_1}}\right|}{\left|x(t)-\sqrt{\frac{c_2}{ac_1}}\right|}\cdot\frac{\left|x(0)-\sqrt{\frac{c_2}{ac_1}}\right|}{\left|x(0)+\sqrt{\frac{c_2}{ac_1}}\right|}\right) = 2\sqrt{ac_1c_2}t, \tag{23}$$

$$\frac{x(t)+\sqrt{\frac{c_2}{ac_1}}}{\sqrt{\frac{c_2}{ac_1}}-x(t)}\cdot\frac{\sqrt{\frac{c_2}{ac_1}}-x(0)}{x(0)+\sqrt{\frac{c_2}{ac_1}}} = e^{2\sqrt{ac_1c_2}t}. \tag{24}$$

Expressing $x(t)$ from the equation concludes the proof. □

Note that formulas (12)–(14) also yield that if $x(0) > 0$ then

$$\lim_{t\to\infty} x(t) = x_2.$$

From the point of view of control, it is important to know when is the difference between the temperature of the collector and the temperature of the storage is sufficiently close to the equilibrium, that is for a given (small) $d > 0$ we need to determine the smallest time instance $t_0 \in [0, \infty[$ for which

$$|x(t_0) - x_2| = d \tag{25}$$

holds. Note, that since x_2 is locally asymptotically stable, for all $t \geq t_0$

$$|x(t) - x_2| < d \tag{26}$$

holds.

Statement 2. *Let $x(0) > 0$ and $d > 0$ be given. Then for the solution of differential equation (7) the relation*

$$|x(t) - x_2| \leq d \tag{27}$$

holds if one of the followings hold:

(a) $x(0) \in]0, x_2[$ *and*

$$t \geq \frac{1}{2\sqrt{ac_1c_2}} \cdot \ln \frac{\left(2 - d\sqrt{\frac{ac_1}{c_2}}\right)\left(\sqrt{\frac{c_2}{c_1}} - \sqrt{a}x(0)\right)}{\left(\sqrt{a}x(0) + \sqrt{\frac{c_2}{c_1}}\right) \cdot d \cdot \sqrt{\frac{ac_1}{c_2}}}, \tag{28}$$

or

(b) $x(0) = x_2$, *or*

(c) $x(0) \in]x_2, \infty[$ *and*

$$t \geq \frac{1}{2\sqrt{ac_1c_2}} \cdot \ln \frac{\left(2 + d\sqrt{\frac{ac_1}{c_2}}\right)\left(\sqrt{a}x(0) - \sqrt{\frac{c_2}{c_1}}\right)}{\left(\sqrt{a}x(0) + \sqrt{\frac{c_2}{c_1}}\right) \cdot d \cdot \sqrt{\frac{ac_1}{c_2}}}. \tag{29}$$

Proof. Case (b) is obvious. Case (a) and Case (c) follows from the rearrangement of inequality (27). □

From the point of view of control it is worth also to know that for a desired equilibrium point $x_2 > 0$ what value of TÉRFOGATÁRAM should be chosen. That is, from the expression for x_2

$$x_2 = \sqrt{\frac{c_2}{ac_1}} \tag{30}$$

we have to express a as a function of x_2. Reorganisation of the above equation yields

$$a = \frac{c_2}{x_2^2 c_1}. \tag{31}$$

2.3 General model for hőcserélő nélküli rendszerre rétegzett tartály esetén, a kollektor hőveszteségének figyelembevételével, arányos szabályozás esetén

Consider the differential equation

$$\dot{T}_c = \frac{A_c \eta_0}{\rho_c c_c V_c} I_c + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \frac{v}{V_c}(T_{cold} - T_c). \tag{32}$$

Arányos szabályozás esetén

$$v = a(T_c - T_{cold}), \tag{33}$$

then

$$\dot{T}_c = -\frac{a}{V_c}(T_c - T_{cold})^2 + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \frac{A_c \eta_0}{\rho_c c_c V_c} I_c. \tag{34}$$

Let us introduce the trivial differential equation for T_{cold}:

$$\dot{T}_{cold} = 0, \tag{35}$$

then we consider the following system

$$\begin{cases} \dot{T}_c = -\frac{a}{V_c}(T_c - T_{cold})^2 + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \frac{A_c \eta_0}{\rho_c c_c V_c} I_c, \\ \dot{T}_{cold} = 0. \end{cases} \tag{36}$$

By taking the difference of the two equations in the previous system we obtain the equation

$$\dot{(T_c - T_{cold})} = -\frac{a}{V_c}(T_c - T_{cold})^2 + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \frac{A_c \eta_0}{\rho_c c_c V_c} I_c. \tag{37}$$

Using the definition for T_{av}, the above equation takes the form

$$\dot{(T_c - T_{cold})} = -\frac{a}{V_c}(T_c - T_{cold})^2 + \frac{U_L A_c}{\rho_c c_c V_c}\left(T_{ce} - \frac{T_c + T_{cold}}{2}\right) + \frac{A_c \eta_0}{\rho_c c_c V_c} I_c, \tag{38}$$

or equivalently

$$\dot{(T_c - T_{cold})} = -\frac{a}{V_c}(T_c - T_{cold})^2 - \frac{U_L A_c}{2\rho_c c_c V_c}(T_c - T_{cold}) + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{cold}) + \frac{A_c \eta_0}{\rho_c c_c V_c} I_c. \tag{39}$$

Let us introduce the new variable

$$y = T_c - T_{cold} \tag{40}$$

and the notations

$$c_3 = \frac{1}{V_c}, \qquad c_4 = \frac{U_L A_c}{2\rho_c c_c V_c}, \qquad c_5 = \frac{A_c \eta_0}{\rho_c c_c V_c} I_c + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{cold}). \tag{41}$$

Note that while c_3 and c_4 are always positive, c_5 is not for all arbitrary parameter values, but in real life applications it usually is, since the specific solar heating system turns on only in the case when $T_{ce} \geq T_{cold}$. Therefore, in this paper we shall assume that $c_5 > 0$. With these notations equation (39) can be written in the form

$$\dot{y} = -ac_3 y^2 - c_4 y + c_5. \tag{42}$$

Since for the quadratic function

$$g(y) = -ac_3 y^2 - c_4 y + c_5 \tag{43}$$

of the right side of equation (42) $c_4^2 + 4ac_3c_5 > 0$ holds, therefore it has two distinct real roots, furthermore they are of opposite sign

$$y_1 = -\frac{c_4 + \sqrt{c_4^2 + 4ac_3c_5}}{2ac_3}, \qquad y_2 = -\frac{c_4 - \sqrt{c_4^2 + 4ac_3c_5}}{2ac_3}, \tag{44}$$

which also means that g can be factorized:

$$g(y) = -ac_3 y^2 - c_4 y + c_5 = -ac_3(y - y_1)(y - y_2). \tag{45}$$

It follows as well that the differential equation (42) has two equilibria, namely y_1 and y_2. It is important to note that only the case $y(0) = T_c(0) - T_{cold}(0) > 0$ is relevant. Observe that y_2 is a locally asymptotically stable equilibrium. For all solution for which $y(0) \in]0, y_2[$ are strictly monotonically increasing, furthermore all solution for which $y(0) \in]y_2, \infty[$ are strictly monotonically decreasing due to the sign of g on these intervals.

IDE TEGYÜNK EGY ÁBRÁT AZ 1D-S FÁZISDIAGRAMMRÓL?

Before stating the main result of this section let us introduce the notation

$$\alpha = y_2 - y_1 = \frac{\sqrt{c_4^2 + 4ac_3c_5}}{ac_3}. \tag{46}$$

Statement 3. *Depending on $y(0) > 0$ the solution of differential equation* (42) *is the following for all $t \in [0, \infty[$:*

(a) if $y(0) \in]0, y_2[$ then

$$y(t) = y_2 - \frac{\alpha}{\frac{y(0)-y_1}{y_2-y(0)} \cdot e^{c_3 \alpha a t} + 1}, \tag{47}$$

(b) if $y(0) = y_2$ then

$$y(t) \equiv y_2, \tag{48}$$

(c) if $y(0) \in]y_2, \infty[$ then

$$y(t) = y_2 + \frac{\alpha}{\frac{y(0)-y_1}{y_2-y(0)} \cdot e^{c_3 \alpha a t} - 1}. \tag{49}$$

Proof. The proof is analogous to that of Statement 1, therefore the details are left to the Reader. □

Next, we shall solve the inequality

$$|y(t) - y_2| \leq d \tag{50}$$

for a given, sufficiently small $d > 0$. Compared to the previous case, here we will need an extra, technical assumption as well.

Statement 4. *Let $y(0) > 0$ and $\alpha > d > 0$ be given. Then for the solution of differential equation* (7) *the relation*

$$|y(t) - y_2| \leq d \tag{51}$$

holds if one of the followings hold:

(a) $y(0) \in]0, y_2[$ and

$$t \geq \frac{1}{c_3 \alpha a} \cdot \ln \frac{(\alpha - d)(y_2 - y(0))}{d(y(0) - y_1)}, \tag{52}$$

or

(b) $y(0) = y_2$, or

(c) $y(0) \in]y_2, \infty[$ *and*

$$t \geq \frac{1}{c_3 \alpha a} \cdot \ln \frac{(\alpha + d)\,(y(0) - y_2)}{d(y(0) - y_1)}. \tag{53}$$

Proof. Case (b) is obvious. Case (a) and Case (c) follows from the rearrangement of inequality (51). □

Remark 1. *Az St. 3-ban és 4-ben szereplő formulák a* $c_4 = \frac{U_L A_c}{\rho_c c_c V_c} = 0$ *választással érvényesek arra az esetre is, amikor elhanyagoljuk a kollektor hőveszteségét.*

From the point of view of control it is worth also to know that for a desired equilibrium point $y_2 > 0$ what value of TÉRFOGATÁRAM should be chosen. That is, from the expression for y_2

$$y_2 = -\frac{c_4 - \sqrt{c_4^2 + 4ac_3c_5}}{2ac_3} \tag{54}$$

we have to express a as a function of y_2. Reorganisation of the above equation gives

$$2ac_3y_2 + c_4 = \sqrt{c_4^2 + 4ac_3c_5}, \tag{55}$$

which, after taking the square of both sides of the equation and simplification leads to

$$a(ac_3y_2^2 + c_4y_2 - c_5) = 0. \tag{56}$$

Since $a > 0$ it follows that

$$a = \frac{c_5 - c_4y_2}{c_3y_2^2}. \tag{57}$$

2.4 General model for hőcserélő nélküli rendszerre rétegzett tartály esetén, a kollektor hőveszteségének figyelembevételével, konstans térfogatáram esetén

Let us again consider differential equation (32) in the case when v is a positive constant.

$$\dot{T}_c = \frac{A_c\eta_0}{\rho_c c_c V_c} I_c + \frac{U_L A_c}{\rho_c c_c V_c}(T_{ce} - T_{av}) + \frac{v}{V_c}(T_{cold} - T_c). \tag{58}$$

Then, using the definition of T_{av} and after reorganisation one can easily see that the equation takes the form

$$\dot{T}_c = -\left(\frac{U_L A_c}{2\rho_c c_c V_c} + \frac{v}{V_c}\right) T_c + \frac{A_c \eta_0}{\rho_c c_c V_c} I_c + \frac{U_L A_c T_{ce}}{\rho_c c_c V_c} + \left(\frac{v}{V_c} - \frac{U_L A_c}{2\rho_c c_c V_c}\right) T_{cold}. \tag{59}$$

Observe that this differential equation is an inhomogeneous linear one with constant coefficients. Let

$$c_6 = \frac{U_L A_c}{2\rho_c c_c V_c} + \frac{v}{V_c}, \qquad c_7 = \frac{A_c \eta_0}{\rho_c c_c V_c} I_c + \frac{U_L A_c T_{ce}}{\rho_c c_c V_c} + \left(\frac{v}{V_c} - \frac{U_L A_c}{2\rho_c c_c V_c}\right) T_{cold}, \tag{60}$$

note that the first constant is always positive, furthermore, in usual real life applications

$$v > \frac{U_L A_c}{2\rho_c c_c} \tag{61}$$

holds. RICSI, EZ NÁLUNK TELJESÜL, DE EZ IGAZ ÁLTALÁBAN IS???? With the introduction of these constants (59) becomes

$$\dot{T}_c = -c_6 T_c + c_7. \tag{62}$$

Statement 5. *The solution of differential equation* (62) *is*

$$T_c(t) = \left(T_c(0) - \frac{c_7}{c_6}\right) e^{-c_6 t} + \frac{c_7}{c_6}. \tag{63}$$

Proof. Application of the constant variation formula concludes the proof. □

Note, that the only (positive) equilibrium point of differential equation (62) is

$$T_{c,eq} = \frac{c_7}{c_6} \tag{64}$$

which is globally asymptotically stable. It means that independently of the initial value of collector's temperature all solutions will tend to $T_{c,eq}$ as t tends to infinity. Furthermore, for all solution for which $T_c(0) \in]0, T_{c,eq}[$ are strictly monotonically increasing and all solution for which $T_c(0) \in]T_{c,eq}, \infty[$ are strictly monotonically decreasing.

Next, we shall solve the inequality

$$|T_c(t) - T_{c,eq}| \leq d \tag{65}$$

for a given, sufficiently small $d > 0$.

Statement 6. *Let $T_c(0) > 0$ be given. Then for the solution of differential equation* (62) *the relation*

$$|T_c(t) - T_{c,eq}| \leq d \tag{66}$$

holds if one of the followings hold:

(a) $T_c(0) \in]0, T_{c,eq}[$ *and*

$$t \geq \frac{1}{c_6} \cdot \ln \frac{-T_c(0) + \frac{c_7}{c_6}}{d}, \tag{67}$$

or

(b) $T_c(0) = T_{c,eq}$, *or*

(c) $T_c(0) \in]T_{c,eq}, \infty[$ *and*

$$t \geq \frac{1}{c_6} \cdot \ln \frac{T_c(0) - \frac{c_7}{c_6}}{d}. \tag{68}$$

Proof. Case (b) is obvious. Case (a) and Case (c) follows from the rearrangement of inequality (66). □

Remark 2. *Az St. 6-ban és 7-ben szereplő formulák a*

$$c_6 = \frac{v}{V_c}$$

választással érvényesek arra az esetre is, amikor elhanyagoljuk a kollektor hőveszteségét.

From the point of view of control it is worth also to know that for a desired equilibrium point $T_{c,eq}$ what value of v should be chosen. That is, from the expression for $T_{c,eq}$

$$T_{c,eq} = \frac{c_7}{c_6} = \frac{\frac{A_c \eta_0}{\rho_c c_c} I_c + \frac{U_L A_c T_{ce}}{\rho_c c_c} + v T_{cold}}{\frac{U_L A_c}{\rho_c c_c} + v} \tag{69}$$

we have to express v as a function of $T_{c,eq}$. Direct calculations give

$$v = \frac{A_c \eta_0 I_c + U_L A_c T_{ce} - U_L A_c T_{c,eq}}{\rho_c c_c (T_{c,eq} - T_{cold})}. \tag{70}$$

GAME-THEORETICAL MODEL FOR SUSTAINABLE USE OF GROUNDWATER IN THE HEAVILY STRESSED SYSTEM OF THE ACQUE ALBULE BASIN (ROME, ITALY)

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X. Mathematical model

X.1. Model building

The drawdown of the water level in a given quarry group is a result of pumping in all quarry groups. Furthermore, the volume of the extracted travertine is considered approximately proportional to the water volume pumped out from the given excavation area. Hence every quarry group is interested in the maximization of the total volume of water pumped in this quarry group. Therefore there is a conflict situation where the application of a *game-theoretical model* is at hand. In addition, there will also be sustainability conditions related to the Acque Albule spring and the Aniene river.

At this point we introduce the usual terms and notation of game theory: The quarry groups are considered *players* (numbered $i = 1, \cdots, 14$). Every player *i* chooses a pumping rate $x_i = Q_i$ as *strategy*. Vector $x = (x_1, \cdots, x_{14})$ called *multi-strategy*, determines the drawdown $\Delta h_i(x_1, \cdots, x_{14})$ in quarry group *i*. If A_i is the excavation area of the latter quarry group, then

$$f_i(x_1, \cdots, x_{14}) = \Delta h_i(x_1, \cdots, x_{14}) A_i \tag{X1}$$

is considered the gain or *payoff function* of player *i*. Here, for the multi-strategy $(x_1, \cdots, x_{14})$ we have

$$0 \leq x_i \leq Q_{i\,max}, \tag{X.2}$$

where $Q_{i\,max}$ is the maximum (possible) pumping rate in the *i*th group.

Now sustainability conditions are described as follows:

The drawdown of water level of the Acque Albule spring and the Aniene river are also dependent on the pumping rates x_i, and we have the following two sustainability constraints for them (in metres):

$$\Delta h_{ab}(x_1, \cdots, x_{14}) = g_1(x_1, \cdots, x_{14}) \leq 1{,}27 \, , \tag{X.3}$$

$$\Delta h_{riv}(x_1, \cdots, x_{14}) = g_2(x_1, \cdots, x_{14}) \leq 3{,}84 \, . \tag{X.4}$$

Now, the strategy choice of the players is limited by these constraints. Therefore we define the *set of admissible multi-strategies*:

$$G = \{(x_1, \cdots, x_{14}) \in [0, Q_{1\,max}] \times \ldots \times [0, Q_{14\,max}] | g_1(x_1, \cdots, x_{14}) \leq 1.27, g_2(x_1, \cdots, x_{14}) \leq 3.84\}$$

Hydrogeological simulations have shown that these constraints really count, since e.g. pumping at maximum rates in all groups is not sustainable.

X.2. Cooperative solutions of a game

The idea of a simultaneous maximization of all payoff functions leads to a constrained *vector (or multi-criterial) optimization problem*

$$F(x) \to \max \quad (x \in G),$$

where $F = (f_1, f_2, \ldots, f_{14})$.

The idea of a *cooperative solution* of the game is based on the concept of *Pareto optimality*. Keeping the above notation also for the general case of *n* players, let *G* be a subset of $\mathbf{R}^n$ and $F = (f_1, f_2, \ldots, f_n)$ a function mapping *G* into $\mathbf{R}^n$.

Point $x^* \in G$ is called *Pareto optimal* for *F* (and the corresponding function value $F(x^*)$ is *called Pareto optimal value*), if there is no other multi-strategy

$$x \in G \text{ such that } f_i(x^*) \leq f_i(x) \ \ (i = 1,2, \cdots, n),$$

and at least for one *i* the inequality is strict. If *G* is interpreted as the set of admissible multi-strategies, $(f_1, f_2, \ldots, f_n)$ are the payoff functions, x^* is also called a *cooperative solution of the n-person game*. Cooperative solution means that there is no other strategy choice that makes one player better off without making some other player worse off.

The set *P* of the above Pareto optimal function values is called the *Pareto frontier of function F* (Blasco *et al.* 2008):

$$P = \{F(x) | x \in G \text{ is Pareto optimal for } F\}.$$

X.3. Scalarization

Let $\boldsymbol{S}^{\mathbf{n}}$ be the interior of the standard simplex in $\mathbf{R}^{n}$, that is, the set of all vectors $(\lambda_1, \dots, \lambda_n)$ with positive coordinates summing to 1. Then it is straightforward to check that for all $\lambda \in \boldsymbol{S}^{n}$, any solution of the scalar optimization problem

$$\lambda_1 f_1(x) + \cdots + \lambda_n f_n(x) \to \max \qquad (x \in G). \tag{X.5}$$

is Pareto optimal for F. In practice, except for some degenerate cases, most Pareto points can be obtained by the above sclarization, see Geoffrion (1968). Using the Pareto points, the Pareto frontier of F can be obviously generated.

From the above construction it is obvious that by scalarization, in general, an infinite number of cooperative solutions can be obtained. In fact, as the illustrative case of Figure X+1.1. b) below shows that the Pareto frontier is the "North-East border" of the range of function *f*.

Considering a game, there should be a reasonable way to single out a unique cooperative solution. In the following, from Kicsiny *et al.* (2022) we adapt the construction of the so-called nearly ideal solution.

X.4. Nearly ideal solution of the game

The following construction was proposed by Salukvadze (1971a,b) for dynamic vector optimization problems, and soon was easily adapted to the static case. First, for every fixed $i = 1,2,\cdots,n,$ we solve the scalar optimization problem

$$f_i(u) \to \max \quad (u \in G) \tag{X.6}$$

$(i = 1,\cdots,n)$, that is, we maximize the payoff of player *i* considering all admissible multi-strategies. Suppose that $u_i^* \in G$ is a solution of problem (X.6), and define $\omega_i = f_i(u_i^*)$. Then vector

$$\Omega = (\omega_1, \dots, \omega_n) \tag{X.7}$$

is called the *ideal value of the game*. Of course, in general, there is no multi-strategy where the ideal value would be attained. Nevertheless, in the Pareto frontier we can seek a point nearest to the ideal value. If for some multi-strategy $u^0 \in G$, we have $F(u^0) \in P$, and $F(u^0)$ is a solution of the optimization problem

$$d(p, \Omega) \to \min \ (p \in P), \tag{X8}$$

then u^0 is called a *nearly ideal solution of the game*. (Here *d* denotes the Euclidean distance.) $F(u^0)$ can be called the *nearly ideal value of the game*.

Remark X.1. We note that in the application to be presented below, the mathematical properties of the functions involved in the model will imply that optimization problems (X.5) and (X.6) have solutions. In the numerical realization of the above game-theoretical model (after discretization), optimization problem (X.8) will also have a solution.

X.5. Data, model fitting and numerical realization

For the definition of the constraints and the payoff functions, we will need the data of Table X.1.

Table X. 1. Some model parameters

Quarry group	$Q_{i\ max}$ (Max pumping rate, m^3/s)	A_i (Area, m^2)
1	0,47	48925
2	0,66	112200
3	0,89	330514
4	0,38	317525
5	0,63	271875
6	0,20	288000
7	0,23	256375
8	0,32	169375
9	0,50	124750
10	1,12	118300
11	0,18	251000
12	0,18	79100
13	0,08	110700
14	0,08	268650

In order to solve the (scalarization of the) vector optimization problem, we need mathematical formulae for the payoff functions f_i, as well as for the constraint functions g_1 and g_2, for which it is sufficient to have formulae for

$\Delta h_i\ (i = 1, \dots, 14), \quad \Delta h_{ab}, \Delta h_{riv},$

the drawdown of water level for each quarry group as well as that of the spring and the river, as functions of the multi-strategy $(x_1, \cdots, x_{14})$. (See formulae (X.2-(X.4))

With the aid of a hydrogeological software VINCENZO, PLEASE, PUT IN ITS NAME, several simulations have been carried out in different scenarios. In these cases the above drawdown values were computed with certain preset multi-strategies (pumping rates). For example, in scenario S1 all groups pump at maximum rate (maximal drawdown), in scenario S0 there is zero pumping (with zero drawdown). The effect of pumping in each single group was assessed, too. In these cases, the pumping rate in each one of the quarry groups was set to be 100, 70 and 50% of its maximum, respectively, while there was no pumping in the other groups. Results of these simulations can be found in the Appendix. VINCENZO, IS THIS OK WITH YOU? OR SHOULD WE PUT IN HERE YOUR SHORT DESCRIPTION OF THE SIMULATION MODEL, AND THE RESULTS AT THE BEGINNING OF THE NEXT SECTION?

X+1. Results

X+1. 1. General results

In order to produce mathematical formulae for the water drawdowns as functions of the multi-strategy, quadratic regression was applied to this set of data. More precisely, we used the following regression function for each index *i*, where

$1 \leq i \leq 14, \quad \text{or} \quad i = ab, \quad \text{or} \quad i = riv.$

$$\Delta h_i(x_1, \cdots, x_{14}) = c^i + b_1^i x_1 + b_2^i x_2 + \cdots + b_{14}^i x_{14} + a_1^i x_1^2 + a_2^i x_2^2 + \cdots + a_{14}^i x_{14}^2. \quad \text{(X+1. 1)}$$

Since we have 44 observations and 29 free coefficients, this kind of regression problem is mathematically correct. Results of the regression can be found in the Appendix. The corresponding correlation coefficients (R values) are mostly very close to 1, which shows a fairly good approximation.

The numerical realization of the Pareto frontier results in a finite "approximation" obtained from a finite number of scalarized optimization problems. In fact, we have to choose a reasonable number of discrete points, which are "quasi-uniformly distributed" in $\boldsymbol{S^n}$. If $r > n$ is a positive integer, then it is obvious to pick all possible *n*-tuples, for which each coordinate is an integer multiple of 1/r. The number of such *n*-tuples is $\binom{r-1}{n-1}$.

The numerical solution of the above constrained scalar optimization problems was carried out with the Matlab software (Etter et al., 2004), particularly using its *fmincon* function. The results are presented in Table X+1. Filling the table, we use easily decodable, less mathematical notation. Firstly, from (X.7) we calculated in the ideal value of the game

$$F_{ideal} = \Omega.$$

Then we generated Pareto optimal solutions in case $r = 20$, that is, for $\binom{19}{13} = 27{,}132$ different λ vectors. From these points we have chosen the *nearly ideal value* of the game, $F_{near_{id}} = F(u^0)$, and the corresponding multi-strategy strat_near_id. Taking uniform weights $\lambda = (\frac{1}{14}, \ldots, \frac{1}{14})$, we determined $F_{\max\_tot}$, the maximal total payoff and the corresponding multi-strategy strat_maxtot. Q_max is the first column of Table X.1, row lambda_near_id indicates the weights of scalarization corresponding to the nearly ideal value. In the last 2 rows there are the payoff values (and the total payoff) of scenario S1 belonging to the (non-admissible) multi-strategy, when all groups pump at maximum rate. Values are calculated with data of simulation and regression, respectively.

Table X+1.1. Results of the game-theoretical model

quarry group	1	2	3	4	5	6	7
F_ideal (m^3)	763670	2192808	6598736	6172856	5871791	4675979	4186094
F_near_id (m^3)	676276	2072242	6357106	5944006	5666533	4482095	3982089
F_maxtot (m^3)	707095	2135435	6413529	6102573	5702328	4611468	4083563
strat_near_id (m^3/s)	0,275	0,598	0,457	0,354	0,400	0,194	0,164
strat_maxtot (m^3/s)	0,324	0,650	0,409	0,380	0,380	0,202	0,195
Q_max (m^3/s)	0,473	0,662	0,893	0,380	0,630	0,202	0,231
lambda_near_id	0,05	0,05	0,05	0,1	0,05	0,05	0,05
payoff_absmax_sim (m^3)	890435	2614260	8114119	7493590	7286250	5731200	5178775
payoff_absmax_reg (m^3)	889479	2610589	8097071	7483192	7272892	5725416	5173992

quarry group	8	9	10	11	12	13	14	sum
F_ideal	2720281	2339733	2300673	2804380	917795	1165468	2994897	45705162
F_near_id	2471170	1968461	1834207	2591351	843301	1058888	2661229	42608955
F_maxtot	2491499	1893157	1534351	2691179	853977	1056236	2598965	42875355
strat_near_id	0,228	0,334	0,781	0,098	0,083	0,052	0,054	
strat_maxtot	0,263	0,375	0,498	0,137	0,106	0,058	0,058	
Q_max	0,315	0,500	1,115	0,180	0,180	0,080	0,080	
lambda_near_id	0,1	0,1	0,15	0,05	0,05	0,05	0,1	
payoff_absmax_sim	3319750	2669650	2531620	3539100	1162770	1461240	3653640	55646399
payoff_absmax_reg	3317266	2667550	2530334	3537392	1162200	1460688	3652095	55580157

X+1. 2. Illustrative case of 2 players

Although there is no appropriate geometric illustration for the case 14 quarry groups (14 players), the fictitious case of two players can be illustrated quite well. In this case only 2 selected groups take part in the game. The others do not, that is, their pumping rates are set to be constant. Now the set of all admissible strategies and the corresponding payoff vectors can be illustrated on 2 dimensional plots.

First, formally let us fix $x_i = 0.7Q_{i\ max}$ for $i \neq 5, 7$ (the pumping rates are considered constant for the other quarry groups), and consider that only players 5 and 7 play. Only pairs (x_5, x_7) are considered multi-strategies. Figure X+1.1.a) shows the set of all admissible multi-strategies (red region).

Without the sustainability conditions this would be a rectangle defined by the respective maximum pumping rates (for players 5 and 7, these equal 0.63 and 0.23 m^3/s, resp., See Table X. 1). In Figure X+1.1.b) the green region represents the set of all payoff vectors belonging to the admissible strategies, in other words, the image of the red region of Figure X+1.1.a) with respect to the vector function F. (It also reflects the nonlinearity of the payoff functions.)

The magenta line shows the Pareto frontier, which typically is the North East segment of the boundary of the green region. On this line the nearly ideal value of the game is indicated by a blue asterisk. This point has minimum Euclidean distance from the ideal value, which is situated outside the green region (indicated by red asterisk).

The vector connecting the red and blue asterisks would be in fact orthogonal to the Pareto frontier if the aspect ratio of the coordinate system were equal to 1.

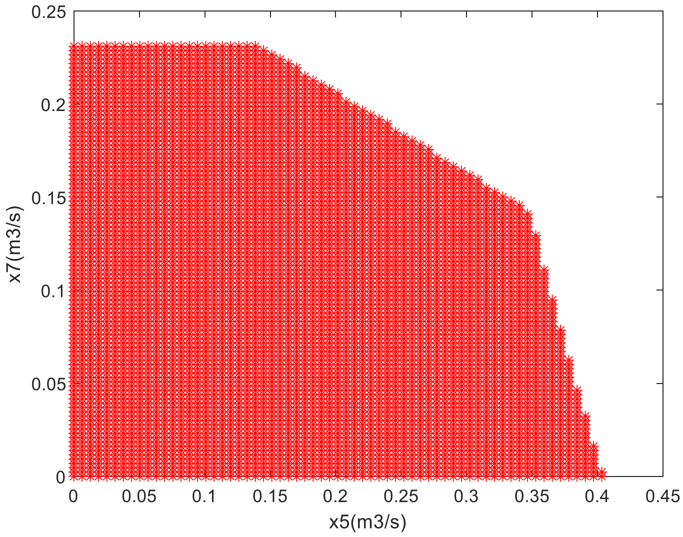

Figure X+1.1. a) Set of admissible multi-strategies, playing only players 5 and 7

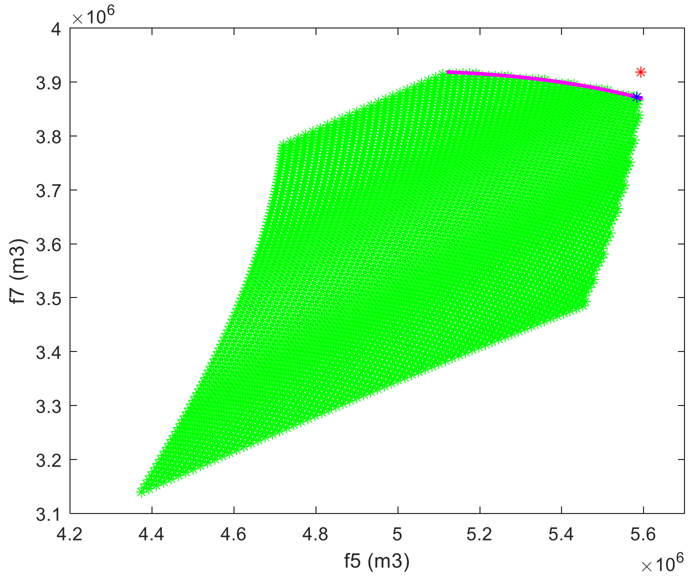

Figure X+1.1. b) Set of all payoff vectors corresponding to the admissible multi-strategies of Figure X+1.1. a)

X+2. Discussion and outlook

Let us first discuss the results of the game model contained in Table X+1.1. First we can check our model whether the obtained numerical results are conform with some intuitive considerations. It is obvious that each coordinate of F_{ideal} must be greater than or equal to the corresponding coordinate of F_{near_id}, which obviously holds in our table. Also notice that the maximal total payoff 42 875 355 is a bit greater than 42608955, the sum of the payoffs of the nearly ideal value.

If the players have an intension to further cooperate, the maximization of the total payoff also may be an issue. Then the total payoff can be redistributed according to an agreement. In Table X+1.1, as expected, the sum of coordinates of $F_{\max\_tot}$ is greater than that of F_{near_id}, and by coordinates, F_{ideal} is greater than or equal to $F_{\max\_tot}$.

For a comparison, in the last 2 rows there are the payoff values (and the total payoffs) of scenario S1 belonging to the (non-admissible) multi-strategy, when all groups would pump at maximum rate. These values were computed in two different ways: based on the results of the hydrogeological simulation (9th row) and by means of regression (10th row). Now the total payoff considering e.g. the simulated value 55 646 399 is substantially greater than that we can achieve under the sustainability constraints (42 875 355). Hence the “price of sustainability” would be around 0,23% reduction of the total exploited travertine 0,23%. Applying our model, the price of sustainability can be estimated for more strict and less strict sustainability constraints.

In the present paper we dealt with cooperative solutions of our game-theoretical model. It might be interesting to find out, what about the non-cooperative solutions. We note that in Kicsiny (2022), thanks to the particular mathematical form of the game constrained by specific sustainability conditions, the obtained cooperative solution also turned out to be non-cooperative solution (Nash equilibrium).

Concerning the modelling methodological aspect of our study, let us observe that further versions and generalizations of the nearly ideal (or Salukvedze) solution of a game could be also applied, see Yu and Leitmann (1974) and LI, Duan (1990). For a somewhat different approach to multi-creterial optimization see Varga (1978).

Finally we note that a continuation of the present research, a model should be developed for the case of the high-season when the Spa is open and pumping, therefore it will be an additional active player in the model.

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Blasco, X., Herrero, J.M., Sanchis, J. and Martínez, M. (2008). A new graphical visualization of *n*-dimensional Pareto front for decision-making in multiobjective optimization. *Information Sciences*, 178(20), 3908-3924.

Etter, D. M., Kuncicky, D., & Moore, H. (2004). Introduction to MATLAB 7. Springer.

Geoffrion, A. M. (1968). Proper efficiency and the theory of vector maximization. *Journal of mathematical analysis and applications*, *22*(3), 618-630.

Kicsiny, R. Piscopo V., Scarelli, A. and Varga, Z. Game-theoretical model for the sustainable use of thermal water resources: the case of Ischia volcanic island (Italy) *Environmental Geochemistry and Health* (2022) 44(7), pp. 2021-2035.

LI, Duan. A new solution approach to Salukvadze's problem. In *1990 American Control Conference*. IEEE, 1990. p. 409-414.

Salukvadze, M. E., Optimization of Vector Functionals, I, Programming of Optimal Trajectories (in Russian), *Avtomatika i Telemekhanika*, No. 8, pp. 5-15. 1971a.

Salukvadze, M. E., Optimization of Vector Functionals, II, The Analytic Construction of Optimal Controls (in Russian), *Avtomatika i Telemekhanika*, No. 9, pp. 5-15, 1971b.

Yu, P. L., and Leitmann, G. Compromise solutions, domination structures, and Salukvadze's solution. *Journal of Optimization Theory and Applications*, (1974) 13, 362-378.

Varga, Z. (1978). Least squares solution for *N*-person multicriteria differential games. *Annales Univ. Sci. Bud., Sectio Mathematica*, XXI, 139-148.