Qualitative properties of the solutions of differential equations

MATH: final report of the project 130513

The aim of the project was to study the qualitative properties of differential equations and differential inequalities. Differential equations are used as models of biological, physical, chemical and economic processes. Their investigations are often motivated by applications.

As a result of our research, we have published 10 papers. According to SJR (Scimago Journal Rankings) 2 papers appeared in international journals of D1, 4 papers in Q1 and 1 paper in Q2 category. In addition, we have published 1 book chapter, 1 journal paper and 1 paper in conference proceedings. Next we briefly describe our most significant results.

Consider the linear delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \geq t_0,$$

where $\tau > 0$ and $p\colon [t_0, \infty) \to [0,\infty)$ is a continuous function. A well-known result in the oscillation theory states that the condition

$$B := \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds \geq \frac{1}{e}$$

is necessary for the oscillation of all solutions. The main result of [1] says that if the coefficient function p is slowly varying at infinity, that is

$$p(t+s) - p(t) \to 0, \qquad t \to \infty,$$

for all $s \in \mathbb{R}$, then $B > 1/e$ implies that all solutions are oscillatory.

In [2] we have studied the delay differential equation

$$x'(t) = \sum_{k=1}^{N} \kappa_k \left(\int_{-\tau}^{0} x(t+s)^{y_k}\, d\mu_k(s) y_k' - x(t)^{y_k} y_k \right)$$

where $\tau, \kappa_k > 0$, the kernels $\mu_k : [-\tau,0] \to [0,\infty)$ are nondecreasing, $\int_{-\tau}^{0} \mu_k(s)\,ds = 1$, $y_k', y_k \in \mathscr{K} \subset \mathbb{Z}_+^d$ and $x^y = \prod_{i=1}^{d} x_i^{y_i}$ whenever $x \in \mathbb{R}_+^d$ and $y \in \mathbb{Z}_+^d$. The above nonlinear differential equation with distributed delay serves as a model in reaction kinetics. In this paper, we have generalized our previous results to equations with distributed delays. In the literature, we were the first to extend the definition of the positive stoichiometric classes from the theory of ordinary differential equations without delays to equations with distributed delays as follows: if $\phi \in C_+ := C([-\tau,0],[0,\infty)^d)$ is an arbitrary nonnegative, continuous initial function on the interval $[-\tau,0]$, then the positive stoichiometric class corresponding to ϕ is defined by

$$\mathcal{S}_\phi = \{\psi \in \ C_+ \mid c_v(\psi) = c_v(\phi) \ \ \forall v \in \mathcal{S}^\perp\} \subset C_+,$$

where $\mathcal{S} = \mathrm{span}\{\ y_k' - y_k \mid 1 \le k \le N\ \}$ and

$$c_v(\psi) := v^T\left[\psi(0) + \sum_{k=1}^{N} \kappa_k \int_{-\tau}^{0}\int_{s}^{0} \psi(u)^{y_k}\, du\ d\mu_k(s)\, y_k\right], \qquad v \in \mathbb{R}^d.$$

A positive equilibrium $\bar{x} \in \mathbb{R}^d_+$ is said to be complex balanced if

$$\sum_{k:\eta=y_k} \kappa_k \bar{x}^{y_k} = \sum_{k:\eta=y_k'} \kappa_k \bar{x}^{y_k}$$

for all $\eta \in \mathcal{K}$. Our main result says that if we assume that the above delay equation has a complex balanced equilibrium, then every equilibrium belonging to the positive stoichiometric class $\mathcal{S}_\phi$ ($\phi \in C_+$) is also complex balanced and semistable. In the proof, we have constructed an appropriate Ljapunov-Krasovski functional and then we have applied LaSalle's invariance principle. Another novelty of the article is the proof of the uniqueness of complex balanced equilibria in the classes $\mathcal{S}_\phi$ ($\phi \in C_+$). We note that our former paper was awarded by Advances in Engineering as a 'Key Scientific Article'.

In [3] we have studied the nonautonomous linear differential equation

$$x' = A(t)x, \qquad t \ge t_0,$$

where $A : [t_0, \infty) \to \mathbb{R}^{d\times d}$ is a bounded, continuous matrix function for which there exists an essentially nonnegative and irreducibile matrix $M \in \mathbb{R}^{d\times d}$ such that $A(t) \ge M$ for all $t \ge t_0$. Under these assumptions, for every $t \ge t_0$, $A(t)$ has a unique nonnegative normalized eigenvector which corresponds to the spectral abscissa and we denote it by $p(t)$. Vector $p(t)$ is called the Perron vector of $A(t)$. In this paper, we have studied the notions of weak and strong ergodicity motivated by applications in population dynamics. The two main results are the weak and the strong ergodic theorems. The former states that for any two solutions x and y starting from positive initial vectors, we have that

$$\frac{x(t)}{\|x(t)\|} - \frac{y(t)}{\|y(t)\|} \longrightarrow 0, \qquad t \to \infty,$$

while the strong ergodic theorem says that if A is uniformly continuous and there exists a positive vector $v \in \mathbb{R}^d_+$ such that

$$p(t) \to v, \qquad t \to \infty,$$

then for every solution x starting from a positive initial vector, we have

$$\frac{x(t)}{\|x(t)\|} \to v, \qquad t \to \infty .$$

In a recent paper, Rodrigues and Solà-Morales presented an example of a nonlinear autonomous difference equation of the form

$$x(t+1) = F(x(t)), \qquad t \in \mathbb{N}$$

such that its trivial solution is exponentially asymptotically stable although the spectral radius $\rho(F'(0))$ of the Fréchet derivative of the nonlinear map $F : X \to X$ on the Banach space X is greater than 1. In [6] we have complemented the results by Rodrigues and Solà-Morales. We have shown that if instead of the notion of exponential asymptotic stability we consider the widely used notion of exponential stability, then the latter is satisfied if and only if $\rho(F'(0)) < 1$. In addition, we have shown that for the above difference equation under the spectral gap condition

$$\sup_{\lambda \in \sigma_1} |\lambda| < \rho(F'(0)), \qquad \sigma_1 := \{\lambda \in \sigma(F'(0)) \mid |\lambda| < \rho(F'(0))\},$$

both stability notions are equivalent to the the condition $\rho(F'(0)) < 1$. Our results provide some important new information to the classical linearized stability principle. The paper was published in the Journal of Differential Equations, the leading journal in our research field.

Integro-differential equations of Volterra type play an important role in biological applications. In [7] we have studied Volterra type difference equations which arise as numerical approximations of integro-differential equations. More precisely, we have studied the difference equation with infinite delay

$$x(t+1) - x(t) = \sum_{j=0}^{\infty} K(j)x(t-j), \qquad t \in \mathbb{Z}_+,$$

where $K\colon \mathbb{Z}_+ \to \mathbb{C}^{n\times n}$. We have shown that if the coefficient matrices are sufficiently small, then the Volterra difference equation with infinite delay is asymptotically equivalent to the ordinary difference equation

$$x(t+1) = Mx(t),$$

where the coefficient matrix M is a solution of the matrix equation

$$M = I + \sum_{j=0}^{\infty} K(j)M^{-j}.$$

In addition, M can be written as a limit $M = \lim_{k\to\infty} M_k$, where

$$M_0 = I, \qquad M_{k+1} = I + \sum_{j=0}^{\infty} K(j)M_k^{-j}, \quad k = 0,1,2,\ldots$$

We have shown that the eigenvalues of M coincide with the dominant characteristic roots of the Volterra difference equation. As a consequence, we have obtained an efficient new method for the numerical approximation of the dominant characteristic roots of the Volterra difference equation.

In [10] we have continued our study of the nonautonomous linear ordinary differential equation

$$x' = A(t)x, \qquad t \geq t_0,$$

where, in addition to the boundedness and continuity of A, we have assumed that, for every $t \geq t_0$, $A(t)$ is a Kirchhoff matrix, that is it is an essentially nonnegative matrix with zero column sums. Equations of this type have applications in reaction kinetics and in the consensus theory of multi-agent systems. The main result of the paper is the following convergence theorem: Suppose that there exists a nonnegative matrix $Q \in \mathbb{R}_+^{d\times d}$ such that

$$Q \leq A(t) - \operatorname{diag} A(t), \qquad t \geq t_0,$$

and the directed graph of Q has a directed spanning tree. Then the convergence of the Perron vectors of the coefficient matrices implies the convergence of all solutions at infinity. More precisely, if $p(t)$ is the Perron vector of $A(t)$ and there exists $v \in \mathbb{R}_+^d$ such that

$$p(t) \to v, \qquad t \to \infty,$$

then for every solution x, we have that

$$\lim_{t\to\infty} x(t) = c(x)v, \qquad \text{where } c(x) = \sum_{i=1}^{d} x_i(t_0).$$

We have applied the result to a chemical reaction model with three components.

Finally, we believe that the book chapter [4] summarizing inequalities for f-divergencies and paper [9] with the refinements of the discrete Hölder's and Minkowski's inequalities are also of interest.

Published papers

1. Á. Garab, M. Pituk, I.P. Stavroulakis: *A sharp oscillation criterion for a linear delay differential equation*, Applied Mathematics Letters 93 (2019), 58-65.
2. Gy. Lipták, M. Pituk, K.M. Hangos: *Modelling and stability analysis of complex balanced kinetic systems with distributed time delays*, Journal of Process Control 84 (2019), 13-23.
3. M. Pituk, C. Pötzsche: *Ergodicity in nonautonomous linear ordinary differential equations*, Journal of Mathematical Analysis and Applications 479 (2019), 1441-1455.
4. S.I. Butt, L. Horváth, G. Pečarić, J. Pečarić: *Improvements of the inequalities for the f-divergence functional with applications to the Zipf-Mandelbrot law*, In: Inequalities and Zipf-Mandelbrot Law, Monographs in Inequalities 15 (Editors: Gilda Pečarić and Josip Pečarić), Element, Zagreb, pages 167-204, 2019.
5. S.P. Pradhan, F. Hartung, J. Turi: *Dynamics in a respiratory control model with two delays*, Applications and Applied Mathematics - An International Journal 14 (2019), 863 -874.
6. Á. Garab, M. Pituk, C. Pötzsche: *Linearized stability in the context of an example by Rodrigues and Solà-Morales*, Journal of Differential Equations 269 (2020), 9838-9845.
7. Á. Fehér, L. Márton, M. Pituk: *Asymptotically ordinary linear Volterra difference equations with infinite delay*, Applied Mathematics and Computation 386 (2020), Paper 125499.
8. M. Pituk: *A note on ergodicity for nonautonomous linear difference equations*, Progress on Difference Equations and Discrete Dynamical Systems (25th ICDEA, London, UK, 2019) Springer Proceedings in Mathematics & Statistics, Vol. 341, pages 37-44, 2020.
9. S.I. Butt, L. Horváth, J. Pečarić: *Cyclic refinements of the discrete Hölder's inequality with applications*, Miskolc Mathematical Notes 21 (2020), 679-687.
10. Á. Garab, M. Pituk: *Convergence in nonautonomous linear differential equations with Kirchhoff coefficients*, Systems & Control Letters Vol. 149 (2021), Paper 104884.