Zárójelentés a 130426 számú pályázathoz, 2018.12.01 – 2022.09.30.

We continued our research in martingale theory and Fourier analysis. We published 29 papers in leading international journals and two books published by Springer. We sdudied different function and martingale spaces, such as wieghted Lebesgue, Musielak-Orlicz, Musielak-Orlicz-Lorentz, mixed-norm Lebesgue spaces and Lebesgue spaces with variable exponents. We investigated also the corresponding martingale Hardy spaces generated by the previous spaces. We proved atomic decompositions, interpolations theorems, duality theorems, martingale inequalitie for these spaces. We also investigated these spaces in Fourier analysis and some related problems.

We considered the A_p weights and weighted martingale Hardy spaces. We presented Doob's maximal inequality and the atomic decomposition of the weighted Hardy spaces. As application, we investigated the convergence of partial sums and Fejér summability of Walsh-Fourier series. We proved that the maximal operator of the Fejér means is bounded from the weighted dyadic Hardy space H_p(w) to L_p(w). This implies almost everywhere convergence of the Fejér means.

We investigated the more general martingale Musielak-Orlicz Hardy spaces. Let (\Omega,F,P) be a probability space and \varphi be a Musielak--Orlicz function. We proved that the Doob maximal operator is bounded on the Musielak—Orlicz space L_\varphi. Using this and extrapolation method, we then established a Fefferman--Stein vector-valued Doob maximal inequality on L_\varphi. As applications, we obtained the dual version of the Doob maximal inequality and the Stein inequality for L_\varphi, which are new even in weighted Orlicz spaces. We established the atomic characterizations of five martingale Musielak--Orlicz Hardy spaces. From these atomic characterizations, we further deduced some martingale inequalities between different martingale Musielak--Orlicz Hardy spaces, which essentially improve the corresponding results in Orlicz space case and are also new even in weighted Orlicz spaces. By establishing the Davis decomposition, we obtained the Burkholder--Davis--Gundy inequality associated with Musielak--Orlicz functions. Finally, using the previous martingale inequalities, we proved that the maximal Fejér operator of the Walsh-Fourier series is bounded from H_\varphi to L_\varphi, which further implies some convergence results of the Fejér means.

For $q\in(0,\infty], we introduced five martingale Musielak--Orlicz--Lorentz Hardy spaces. We proved that these new spaces have some important features such as atomic characterizations, the boundedness of σ-sublinear operators, and martingale inequalities. This new scale of martingale Hardy spaces requires the introduction of the Musielak--Orlicz--Lorentz space

$L^{\varphi,q}(\Omega)$. In particular, we have shown that this Lorentz type space has some fundamental properties including the completeness, the convergence, real interpolations, and the Fefferman--Stein vector-valued inequality for the Doob maximal operator. As applications, we proved that the maximal Fej\'er operator is bounded from the martingale Musielak--Orlicz--Lorentz Hardy space $H_{\varphi,q}[0,1)$ to $L^{\varphi,q}[0,1)$, which further implies some convergence results of the Fej\'er means. Moreover, all the above results are new even for Musielak--Orlicz functions with particular structure such as weight, weight Orlicz, and double phase-growth. One of the main approach used in this topic can be viewed as a combination of the stopping time argument in probability theory and the real-variable technique of function spaces in harmonic analysis. We also characterized the interpolation spaces and the dual spaces of these spaces.

Next, we investigated Lebesgue, Lorentz, Hardy and Hardy-Lorentz spaces with variable exponents. Let p(.) be a variable exponent function satisfying the globally log-Hölder continuous condition and 0<q<=\infty. Let L_{p(.)}, and L_{p(.),q} be the variable Lebesgue- and Lorentz spaces. We studied five different types of variable Hardy spaces H_{p(.)} and five different types of variable Hardy-Lorentz spaces H_{p(.),q}. These spaces generalize the well-known spaces with constant p's. In a very long paper with 67 pages, we proved the basic results about martingale Hardy spaces with variable exponents. We proved their atomic decompositions when each σ-algebra $\mathcal F_n$ is generated by countably many atoms. Martingale inequalities and the relation of the different martingale Hardy spaces are proved as application of the atomic decomposition. In order to get these results, we introduced a new condition to replace (generalize) the so-called log-Holder continuity condition used in harmonic analysis. Some applications in Fourier analysis are given by use of the previous results. We generalize the classical results and show that the partial sums of the Walsh-Fourier series converge to the function in norm if $f \in L_{p(\cdot)}$ or $f \in L_{p(\cdot),q}$ and $p_->1$. The boundedness of the maximal Fejér operator on $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ is proved whenever $p_->1/2$ and the condition $\frac{1}{p_-}-\frac{1}{p_+} <1$ hold. It is surprising that this last condition does not appear for trigonometric Fourier series. One of the key points of the proof is that we introduce two new dyadic maximal operators and prove their boundedness on $L_{p(\cdot)}$ with $p_->1$. The method we used to prove these results is new even in the classical case. As a consequence, we obtain theorems about almost everywhere and norm convergence of the Fejér means. The new dyadic maximal operators just mentioned are considered in a separate paper, too. In another paper, we proved the Burkholder-Davis-Gundy inequality for $1 \leq p_- \leq p_+ < \infty$, which is of great importance. The interpolation spaces of variable Hardy spaces $H_{p(\cdot)}$ are characterized.

We also proved the boundedness of the Cesaro and Riesz maximal operators from H_{p(.)} to L_{p(.)} and from H_{p(.),q} to L_{p(.),q}. As a consequence, we got theorems about almost everywhere and norm convergence of the Cesaro and Riesz means. We presented essential differences between the trigonometric- and Walsh-Fourier series.

We gave a second atomic decomposition for the variable Hardy-Lorentz spaces $H_{p(\cdot),q}$ via simple L_r-atoms $(1<r \leq \infty)$. Using this atomic decomposition,

we considered the dual spaces of $H_{p(\cdot),q}$ for the case $0<p(\cdot)\leq 1$, $0<q\leq 1$, and $0<p(\cdot)<2$, $1<q<\infty$ respectively, and proved that they are equivalent to the BMO spaces with variable exponent. Furthermore, we also obtained several John-Nirenberg theorems based on the dual results.

We introduced a new type of dyadic maximal operators and proved that under the log-Hölder continuity condition of the variable exponent $p(\cdot)$, it is bounded on $L_{p(\cdot)}$ if $1<p_-\leq p_+\leq \infty$. Moreover, the space generated by the $L_{p(\cdot)}$-norm (resp. the $L_{p(\cdot),q}$-norm) of the maximal operator is equivalent to the Hardy space $H_{p(\cdot)}$ (resp. to the Hardy-Lorentz space $H_{p(\cdot),q}$). As special cases, our maximal operator contains the usual dyadic maximal operator and four other maximal operators investigated in the literature.

We studied also the periodic variable Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{T}^d)$ and $H_{p(\cdot),q}(\mathbb{T}^d)$ and proved their atomic decompositions. A general summability method, the so called θ-summability is considered for multi-dimensional Fourier series. Under some conditions on θ, it is proved that the maximal operator of the θ-means is bounded from $H_{p(\cdot)}(\mathbb{T}^d)$ to $L_{p(\cdot)}(\mathbb{T}^d)$ and from $H_{p(\cdot),q}(\mathbb{T}^d)$ to $L_{p(\cdot),q}(\mathbb{T}^d)$. This implies some norm and almost everywhere convergence results for the summability means. The Riesz, Bochner-Riesz, Weierstrass, Picard and Bessel summations are investigated as special cases.

We considered the martingale Hardy spaces defined with the help of the mixed $L_{\pv}$-norm. Five mixed martingale Hardy spaces are investigated. Several results are proved for these spaces, like atomic decompositions, Doob's inequality, boundedness, martingale inequalities and the generalization of the well-known Burkholder-Davis-Gundy inequality. Using atomic decomposition and Doob's inequality, some applications are shown in Fourier-analysis, such as the uniform boundedness of the partial sums of the Walsh-Fourier series on $L_{\pv}$ $(1 < \pv < \infty)$ and the boundedness of the Fejér maximal operator from $H_{\pv}$ to $L_{\pv}$ $(1/2 < \pv < \infty)$. As a consequence of the boundedness, we got some almost everywhere and norm convergence results. We proved another atomic decomposition for mixed norm martingale Hardy spaces via simple atoms. The dual spaces of the mixed-norm martingale Hardy spaces are given as the mixed-norm $BMO_{\vec{r}}(\vec{\alpha})$ spaces. This implies the John-Nirenberg inequality $BMO_{1}(\vec{\alpha}) \sim BMO_{\vec{r}}(\vec{\alpha})$ for $1<\vec{r}<\infty$. These results generalize the well known classical results for constants p and r.

We considered the Cesaro and Riesz means with varying parameters, which is essentially different from the constant parameter case. We proved that the maximal operator of subsequences of the Cesaro and Riesz means with varying parameters is bounded from the dyadic Hardy space H_p to L_p. This implies an almost everywhere convergence for the subsequences of the summability means. These results are generalized for multi-dimensional functions, too.

We published two books at Springer. In the first book (F. Weisz: Lebesgue Points and Summability of Higher Dimensional Fourier Series. Applied and Numerical Harmonic Analysis. Springer, Birkh\"auser, Basel. 2021, 303 pp.) and in three other papers, we investigated some generalizations of Lebesgue points for higher dimensional functions. We gave different generalizations of the classical Lebesgue's theorem to multi-dimensional functions. We investigated different types of Lebesgue points and proved that different types of the Ces\`aro means of the Fourier series of the multi-dimensional integrable function converge to f at each Lebesgue point.

In the second monograph (L. E. Persson and G. Tephnadze and F. Weisz: Martingale Hardy Spaces and Summability of Vilenkin-Fourier Series. Springer, Birkh\"auser, Basel. 2022, 650 pp.), we gave a systematic study of the theory of Vilenkin-Fourier series in more than 650 pages. It includes all necessary theorems on martingale Hardy spaces as well as a new proof for the famous Carleson’s result about the almost everywhere convergence of Vienkin-Fourier series of f \In L_p (1<p<\intfy) and a counterexample for p=1. We investigated also several summability methods for Vilenkin-Fourier series.

Conference talks:

K. Szarvas: Mixed Hardy spaces and martingale inequalities. Conference on Harmonic Analysis and Related Fields, Visegrád, June 11-13, 2019.

F. Weisz: Variable spaces, martingales and applications in Fourier analysis. Matematikai Tudományok Osztálya Fourier analysis and applications című tudományos ülése, May 15, 2019.

F. Weisz: Generalization of Lebesgue points and Walsh-Lebesgue points. Conference on Harmonic Analysis and Related Fields, Visegrád, June 11-13, 2019.

F. Weisz: Laudation to Ferenc Schipp (The scientific work of Prof. Ferenc Schipp). Conference on Harmonic Analysis and Related Fields, Visegrád, June 11-13, 2019.

F. Weisz: Hardy spaces spaces with variable exponents and Fourier analysis. Barcelona Analysis Conference, June 25-28, 2019.

F. Weisz: Hardy spaces with variable exponents and applications in summability. X Jaen Conference on Approximation, June 30-July 05, Ubeda, Jaén, Spain, 2019.

F. Weisz: Variable Hardy spaces and Fourier analysis. IWOTA 2019, International Workshop on Operator Theory and its Applications, Lisbon, Portugal, July 22-26, 2019.

F. Weisz: Lebesgue points of two-dimensional functions and summability. City University of Hongkong, China 2019.

F. Weisz: Summability and Lebesgue points of two-dimensional Fourier transforms. University of Macau, China 2019.

F. Weisz: Hardy spaces with variable exponents and applications in Fourier analysis. International Conference on Function Spaces and Geometric Analysis and Their Applications, September 30-October 04, Nankai University, Tianjin, China 2019.

F. Weisz: The generalizations of Lebesgue points and summability. Normal University, Beijing, China 2019.

F. Weisz: Higher dimensional summability and Lebesgue points. Seminar on Analysis, Differential Equations and Mathematical Physics, Southern Federal University, Regional Mathematical Center, Rostov-on-Don, Russia 2021.

F. Weisz: Hardy spaces with variable exponents in Fourier analysis. Applied Harmonic Analysis and Friends, June 19th - 25th, Strobl, Austria, 2022.

F. Weisz: Hardy spaces with variable exponents and maximal operators. Functional Analysis, Approximation Theory and Numerical Analysis, Matera, Italy, July 5-8, 2022.