Final report of KH 130371

Zoltán Füredi*

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Abstract

Furedi and his group members has maintained a very extensive research activity, published several results. A number of manuscripts completed in this project, most of them have already been appeared, or accepted. Many of them in the most prestigious journals of the field like Journal of Combinatorial Theory, the SIAM J. Discrete Math., European J. Combinatorics, J. Graph Theory. A few more are in preparation. In three years 63 papers are listed, an exceptionally large number in mathematics, in all of them the NKFIH are acknowledged and are Open Access.

Most results concerned with the newest topics in Turán theory, namely about the size (or density) of ordered graphs, hypergraphs, matrices and geometric arrangements. One of the most interesting results states (No 3 among "Válogatott közlemények") that every geometric hypergraph contains a dense, well structured subsystem. This was applied to crack a number of computational geometry questions.

1 Aims, changes

1.1 Title of the project. Structure and stability of low density combinatorial systems
Stabilitás és struktúra kis sűrűségű kombinatorikus rendszerekben

1.2 Aims. To continue research in the most successful field of Hungarian mathematics, i.e., in Combinatorics. One of the newest areas here the investigation of sparse combinatorial structures. We proposed three areas: 1. Turán type problems, esp. Berge hypergraphs, 2. Turán type problems for very thin graphs, like in the linear range, 3. to investigate r-uniform hypergraphs in the Erdos-Ko-Rado range.

There were 60+ manuscripts written, partly because the project has been extended from 2 years to 3 and half. Below we give some details in some half a dozen outstanding achievements, all of those published in Q1 journals (actually these are even D1 publications).

*Alfréd Rényi Institute of Mathematics, P.O. Box 127, Budapest, Hungary, H-1364. `furedi.zoltan@renyi.hu`

1.3 Changes on budget. We seriously underspent the original budget. Because of the pandemic all travels and invitations had been cancelled. Instead of the conference what we had planned to organize we supported the Renyi 100 meeting, its topic was closely related to ours (This change was done by the prior permission of the NKFIH office). Also (with permission of NKFIH) we were able to hire 3 young scientists, two of them MA students (Imolay András, Schweitzer Ádám) who worked on "Product properties on sparse graphs". They already had a manuscript about this and we will complete a further work in the near future.

2 Avoiding long Berge cycles

This Section is based on a paper by *Furedi Zoltan, Kostochka Alexandr, and Luo Ruth* appeared in Journal of Combinatorial Theory B 137: pp. 55–64, 2019.

Let $n \geq k \geq r+3$ and H be an n-vertex r-uniform hypergraph. We show that if $|H| > \frac{n-1}{k-2}\binom{k-1}{r}$ then it contains a Berge **cycle** of length at least k. This bound is tight when $k-2$ divides $n-1$. We also show that the bound is attained only for connected r-uniform hypergraphs in which every block is the complete hypergraph.

This is one of the most heavily investigated field of Turán graph and hypergraph theory recently. This kind of Turán type problems are investigated by many researchers, at least 30 manuscripts came out last year, and solving the case of Berge cycles is one of the outstanding achievements. Some of our results were independently discovered by Győri, Salia and Zamora [12]. They determined the maximum number of hyperedges in an n-vertex, connected, r-uniform hypergraph that does not contain a Berge-**path** of length k provided k is large enough compared to r. They also determined the unique extremal hypergraph H_1. Now we prove a stability version of this result by presenting another construction H_2 and showing that any n-vertex, connected, r-uniform hypergraph without a Berge-path of length k, that contains more hyperedges than the number of hyperedges in H_2 then it must be a subhypergraph of the extremal hypergraph H_1, provided k is large enough compared to r. Overall, stability results in some sense are more important and more dependable than the simple extremal questions.

3 Stability for maximal F-free graphs

This Section is based on a paper by *Gerbner Dániel* which appeared in Graphs Combin. 37 (2021), no. 6, 2571–2580.

Saturation questions: These are many different, important direction of generalizing Turán type problems (their history go back to Erdős, Dirac and others from the 1960's). One of the most difficult part of extremal combinatorics, the usual answers are frequently not monotone, neither natural limit exist. We showed that certain classes of these questions can be handled with newer tools of combinatorics (like the sparse Szemerédi regularity).

In this field Popielarz, Sahasrabuddhe and Snyder [17] proved that maximal $Kr+1$-free graphs with

$\frac{r-1}{2r}n^2 - o(n^{r+1/r})$ edges contain a complete r-partite subgraph on $n - o(n)$ vertices. This was very recently extended to odd cycles in place of K_3 by Wang, Wang, Yang and Yuan [22]. Gerbner further extended it to some other 3-chromatic graphs, and obtained other stability results.

4 A manuscript in the American Mathematical Monthly

This Section is based on a paper by *Furedi Zoltan, Gyarfas Andras:* An Extension of Mantel's Theorem to k-Graphs which appeared in American Mathematical Monthly 127: (3) pp. 263-268., 2020.

This journal has a kind of role as "Nature" for the physicist. It publishes only very well-written, extremely interesting papers which are read by a large circle of mathematicians. Its acceptance rate is less than 1 among 100 ! Its circulation is at least 20–50 times larger than any other mathematical journal. So it was a great success for us that they published a Turán type result for triple systems. The result itself (by our opinion) is not that difficult but nice, with a clever proof. We also pointed out many connections to other fields of mathematics and proposed several problems.

This is a short expository paper (but also containing original results) concerning the linear Turán number of the k-fan. By Mantel's theorem a triangle-free graph on n points has at most $n^2/4$ edges. A linear k-graph is a set of points together with some k-element subsets, called edges, such that any two edges intersect in at most one point. The k-graph Fan^k consists of k edges that pairwise intersect exactly in one point v, plus one more edge intersecting each of these edges in a point different from v. We extend Mantel's theorem as follows: Fan-free linear k-graphs on n points have at most n^2/k^2 edges. This extension nicely illustrates the difficulties of hypergraph Turán problems. The determination of the case of equality leads to transversal designs on n points with k groups — for $k = 3$ these are equivalent to Latin squares. However, in contrast to the graph case, new structures and open problems emerge when n is not divisible by k.

5 Sidon sets

This Section is based on a paper by *J. Balogh, Z. Füredi, and Souktik Roy:* An upper bound on the size of Sidon sets, which has been accepted in Amer. Math. Monthly, a 10 page manuscript.

A set of numbers A is a *Sidon set*, or alternately a B_2*-set*, if $a + b = c + d$, $a, b, c, d \in A$ imply $\{a, b\} = \{c, d\}$, i.e., all pairwise sums are distinct. Let $S(n)$ denote the maximum size of a Sidon set can have when $A \subset \{1, 2, \ldots, n\} =: [n]$. It is obvious that $S(n) \leq 2\sqrt{n}$ since $\binom{|A|+1}{2} \leq 2n - 1$. In 1941, Erdős and Turán [7] observed that a result of Singer (1938) implies that $S(n) > \sqrt{n}$ infinitely many times. They also proved but did not state in that form, which was done much later by Cilleruelo [5], that

$$S(n) < n^{1/2} + n^{1/4} + \frac{1}{2}.$$

Erdős offered \$500 for a proof or disproof that for every $\varepsilon > 0$ the equality $S(n) < \sqrt{n} + o(n^{\varepsilon})$ holds. Combining two classical proofs (by Lindström's argument [14] and Ruzsa [18]) we decrease the gap between the upper and lower bounds by 0.2% for infinitely many values of n. We show that the maximum size of a Sidon set of $\{1, 2, \ldots, n\}$ is at most $\sqrt{n} + 0.998n^{1/4}$ for n sufficiently large.

6 Hypergraphs without exponents

This Section is based on a paper by *Zoltán Füredi and Dániel Gerbner* appeared in J. Combin. Theory Ser. A 184 (2021), Paper No. 105517, 9 pp.

It is a 60 years old problem of Erdos and Simonovits [6] whether the order of magnitude of Turán functions are always polymonial. We say that a function $f : \mathbf{N} \to \mathbf{R}$ has no exponent if there is no real α such that $f(n) = \Theta(n^\alpha)$. In other words, the order of magnitude of f is not a polynomial. For triple systems in the famous $(6, 3)$ theorem Ruzsa and Szemerédi [19] gave a set of forbidden classes whose growth function had order of magnitude $o(n^2)$ (smaller than n^2) but larger than any n^{2-c}. The excluded class can be reduced to two triple systems (linear triangle, $123, 345, 561$ and intersection of size two, $123, 124$). Erdős asked for a one member example. His question was answered by Frankl and Furedi [8] by a 5-uniform example. Since the results of Conlon and Bukh [4] there is a renewed interest in this field. In this paper Furedi and Gerbner gave an example for every r at least 5. The proof is a combination of the algebraic method and the most advanced version of the use of quasirandom hypergraphs (namely the hypergraph regularity lemma).

7 A splitting theorem for ordered hypergraphs

This Section is based on a paper by *Füredi Z., Jiang T., Kostochka A., Mubayi D., Verstraëte J.:* Partitioning ordered hypergraphs which appeared in Journal of Combinatorial Theory A 177: 105300, 2021.

An ordered r-graph is an r-uniform hypergraph whose vertex set is linearly ordered; it is called interval r-partite if there are r consecutive intervals such that each edge has one point in each interval. The basic observation of Erdős and Kleitman concerning large r-partite subgraphs in r-graphs cannot be extended to the setting of interval r-partite subgraphs in ordered hypergraphs. However, we prove that one can obtain a dense subgraph that has the following similar structure. For $2 \leq k \leq r$, an ordered r-graph is interval k-partite if there exist at least k consecutive intervals in the ordering such that every edge has nonempty intersection with each interval. Our main result states that for each $\alpha > k-1$, every n-vertex ordered r-graph with $d \times n^\alpha$ edges has for some $m \leq n$ an m-vertex interval k-partite subgraph with $\Omega(d \times m^\alpha)$ edges. The restriction $\alpha > k - 1$ is sharp. This theorem has applications to several extremal problems for ordered hypergraphs. The investigation of extremal ordered structures is in the center of studies nowdays (one of ICM speakers, namely G. Tardos [20], talked about these) and our structure theorem might have plenty of further applications.

8 Extremal problems for pairs of triangles

This Section is based on a paper by *Zoltan Furedi, Dhruv Mubayi, Jason O'Neill, Jacques Verstraete* appeared in J. Combin. Theory Ser. B 155 (2022), 83–110., 2022.

A *convex geometric hypergraph* or cgh consists of a family of subsets of a strictly convex set of points in the plane. There are eight different intersection patterns of pairs of triangles in convex triangle systems, depicted below:

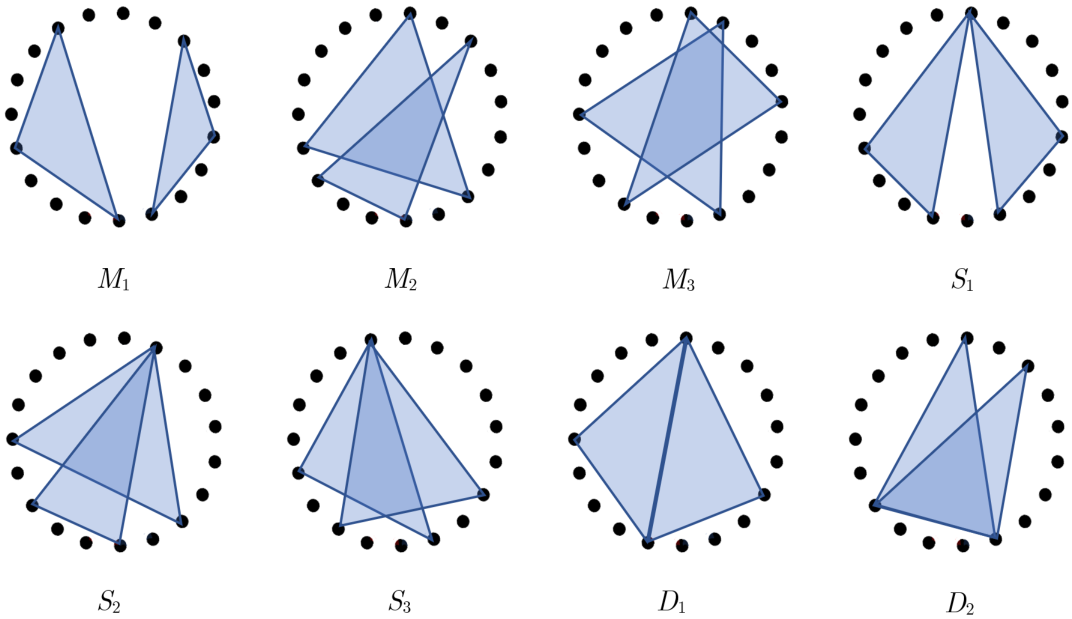


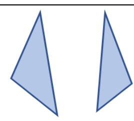
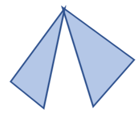
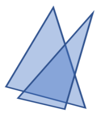
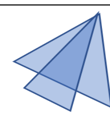
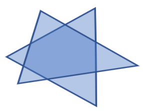
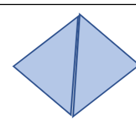
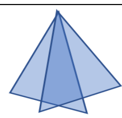
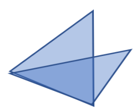
Figure 1: The eight types of triangle pairs in convex triangle systems

We refer to a set of triangles from a set Ω_n of n vertices of a regular n-gon as a *convex triangle system*. It is convenient also to refer to this as a *convex geometric hypergraph* or *cgh*, where the triangles are considered as triples in $\binom{\Omega_n}{3}$, and the vertices of Ω_n are cyclically ordered in the clockwise direction, say $v_0 < v_1 < \cdots < v_{n-1} < v_0$. In this case, we consider the subscripts modulo n. A cgh F is *contained* in a cgh H if there is an injection from $V(F)$ to $V(H)$ preserving the cyclic ordering of the vertices and preserving edges, and we say that H is F*-free* if H does not contain F as a subhypergraph. The extremal function $\mathrm{ex}_{\circlearrowright}(n, F)$ denotes the maximum number of edges in an F-free cgh on Ω_n.

For all of these eight configurations, Braß [3] has shown the extremal function for convex triangle systems is either $\Theta(n^2)$ or $\Theta(n^3)$; the latter arises precisely when the two triangles have no common interior point. Aronov, Dujmović, Morin, Ooms and da Silveira [2] extensively studied cghs which avoid combinations of the configurations in Figure 1, and determined many of the order of magnitudes of the associated extremal numbers. Define

$$\triangle(n) = \begin{cases} \dfrac{n(n-1)(n+1)}{24} & \text{if } n \text{ is odd} \\ \dfrac{n(n-2)(n+2)}{24} & \text{if } n \text{ is even.} \end{cases}$$

We determine the extremal functions exactly for seven of the eight configurations. Our results for $\mathrm{ex}_{\circlearrowleft}(n, F)$ in this paper is summarized in the following table. For S_2 and D_2, we only have bounds on the extremal function, and write $[a, b]$ in the table to denote $a \leq \mathrm{ex}_{\circlearrowleft}(n, F) \leq b$.

F	$\mathrm{ex}_{\circlearrowleft}(n, F)$	F	Bounds on $\mathrm{ex}_{\circlearrowleft}(n, F)$
 M_1	$\triangle(n) + \frac{n(n-3)}{2}$	 S_1	$\triangle(n) + \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor$
 M_2	$\binom{n}{2} - 2$	 S_2	$[\lfloor \frac{n^2}{4} \rfloor - 1, \frac{23n^2}{64}]$
 M_3	$\binom{n}{3} - \binom{n-3}{3}$	 D_1	$\triangle(n)$
 S_3	$\frac{n(n-2)}{2}$ for n even	 D_2	$\frac{2n^2 - 3n}{9}$ for $n \equiv 6 \mod 9$

We also investigate a more general context of triangle systems where the n points are not necessarily convex. In this setting, D_1 denotes two triangles on opposite sides of a line and sharing a side – tangent triangles – and S_1 denotes two triangles intersecting in exactly one vertex – touching triangles – whereas M_1 denotes two triangles sharing no points – separated triangles. We have

Let $F \in \{M_1, D_1, S_1\}$, and let $\mathcal{T}$ be an n-point triangle system of maximum size not containing F. Then

$$|\mathcal{T}| = \begin{cases} \triangle(n) & \text{if } F = D_1 \\ \triangle(n) + \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor & \text{if } F = S_1 \\ \triangle(n) + \Theta(n^2) & \text{if } F = M_1. \end{cases}$$

The additive term of order n^2 for the case of M_1 arises from a geometric theorem of Valtr [21] on *avoiding line segments* in the plane. Thus we answer a question of P. Frankl, Holmsen and Kupavskii [9] who proved $\mathrm{ex}_{\circlearrowleft}(n, \{D_1, M_1, S_1\}) = \triangle(n)$.

For the above configurations $F \in \{D_1, S_1, M_1\}$, the extremal functions $\mathrm{ex}_{\circlearrowleft}(n, F)$ and for the planer triangle systems, $\mathrm{ex}(P, F)$, are equal (almost equal). This is quite exceptional, for most configurations F the non-convex case is much more complex. E.g., one can find a self-intersecting path P_3 of length three in a convex geometric graph with $\Omega(n)$ edges, while for the general not necessarily convex case Pach, Pinchasi, Tardos, and Tóth [15] showed that $\max \mathrm{ex}(P_3, F) = \Omega(n \log n)$.

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