Final Report

1. **Multivariate L^p Bernstein-Markov type polynomial inequalities.** This story started about 170 years ago initiated by a question of Mendeleev and eventually resulted in hundreds of papers with numerous important applications in Approximation Theory and Analysis in general.

The classical Bernstein and Markov inequalities for univariate algebraic polynomials p_n of degree $\leq n$ give the following sharp upper bounds for their derivatives in the uniform norm:

$$\|\sqrt{1-x^2}p_n'(x)\|_{C[-1,1]} \leq n\|p_n\|_{C[-1,1]} \tag{1}$$

and

$$\|p_n'\|_{C[-1,1]} \leq n^2\|p_n\|_{C[-1,1]}. \tag{2}$$

Above inequalities and their numerous generalizations play crucial role in various branches of analysis and approximation theory, it would be difficult to overstate their importance to the field. With respect to their widespread application we would like to point out just three major areas where they are widely used:

A. Inverse Bernstein-Jackson type theorems on the rate of best approximation

B. Marcinkiewicz-Zygmund type results on discretization of norms

C. Study of the rate of Christoffel functions in the theory of orthogonal polynomials

It is considerably harder to establish exact explicit Bernstein-Markov type inequalities similar to (1) and (2) in case of the L^2 norm. In fact even in the univariate case this was accomplished only for certain weights w.

Now let us turn our attention to the fascinating case of the unit ball $B^d := \{x \in \mathbb{R}^d : |x| \leq 1\}$.

We established sharp Bernstein-Markov type inequalities for $L^2(B^d)$. Given any $\mu > -1$ let us consider the weight $w_\mu(x) := (1-|x|^2)^{\frac{\mu}{2}}$. Then the solution of the L^2 Bernstein problem on B^d consists in finding the quantity

$$B_n^d(w_\mu) := \sup_{p \in P_n^d \setminus \{0\}} \frac{\|(1-|x|^2)^{\frac{\mu+1}{2}} Dp\|_{L^2(B^d)}}{\|(1-|x|^2)^{\frac{\mu}{2}} p\|_{L^2(B^d)}}. \tag{3}$$

It turns out that the exact values of $B_n^d(w_\mu, 2)$ can be determined using the classical Jacobi polynomials $J_m^{\alpha,\beta}(t)$.

Theorem 1. *Let $\mu > -1, d \geq 1, n \in \mathbb{N}$. Then*

$$B_n^d(w_\mu) = \begin{cases} \sqrt{n(n+d+2\mu)}, & if\ n\ is\ even, \\ \sqrt{n(n+d+2\mu)-d+1}, & if\ n\ is\ odd. \end{cases} \tag{4}$$

Moreover, the upper bound (4) is attained if and only if $p(x) = cJ_{\frac{n}{2}}^{(\mu,\frac{d}{2}-1)}(2|x|^2-1)$, even n, and $p(x) = J_{\frac{n-1}{2}}^{(\mu,\frac{d}{2})}(2|x|^2-1)(a_1x_1 + ... + a_dx_d)$, $\forall a_j \in \mathbb{R}$, odd n.

Theorem 1 presents a rare occasion when the exact solution of the multivariate L^2 Bernstein-Markov type problem can be found explicitly. When $d > 1$ we found the solution of the L^2 Markov problem for the **homogeneous polynomials** H_n^d on the unit sphere S^{d-1}. As usual, H_n^d stands for the space of *homogeneous polynomials* of degree n in d variables.

Theorem 2. *Let $d \geq 2, n \in \mathbb{N}$. Then for any $h_n \in H_n^d$ we have the sharp estimates*

$$\xi_n \|h_n\|_{L^2(S^{d-1})} \leq \|Dh_n\|_{L^2(S^{d-1})} \leq \sqrt{n(2n+d-2)}\|h_n\|_{L^2(S^{d-1})} \tag{5}$$

with $\xi_n = n$ if n is even and $\xi_n = \sqrt{n^2+d-1}$ if n is odd.

Moreover, the upper bound is attained if and only if h_n is a spherical harmonic polynomial of order n, while the lower bound is attained if and only if $h_n(x) = c|x|^n, \forall c \in \mathbb{R}$ or $h_n(x) = |x|^{n-1}q(x), \forall q \in H_1^d$ when n is even or odd, respectively.

It is well known that for Lip1 and thus, in particular convex domains the upper bound of order n^2 for $L^p, 1 \leq p \leq \infty$ Markov inequality extends for multivariate polynomials. However, when the underlying domain is cuspidal the order of magnitude can change drastically. In particular, for Lipγ, $0 < \gamma < 1$ cuspidal domains in $\mathbb{R}^d$ the sharp order in L^∞ Markov inequality is known to be $n^{\frac{2}{\gamma}}$. Typically multivariate Markov-type inequalities in L^∞ norm are proved by inscribing suitable polynomial curves into the domain and reducing the problem to the univariate setting on these curves. In case of $L^p, 1 \leq p < \infty$ norm this reduction of dimension technique does not work and more delicate considerations are needed.

Let $K \subset \mathbb{R}^d$ be a compact set with nonempty interior. We denote by

$$B(a,r) := \{x \in \mathbb{R}^d : |x-a| \leq r\} \subset \mathbb{R}^d$$

the closed ball of radius r and center a, and $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere. For any $r > 0, a \in \mathbb{R}^d$ and $u \in S^{d-1}$ the cylinder $L_a(r,u)$ of radius $r > 0$, center a and axis u is given by

$$L_a(r,u) := \{x \in \mathbb{R}^d : |x-a|^2 \leq r^2 + \langle x-a, u\rangle^2\}.$$

Furthermore, $l_x(u)$ will denote the line in $\mathbb{R}^d$ in direction $u \in S^{d-1}$ through point $x \in \mathbb{R}^d$.

Definition. *K is called a **graph domain** with respect to the cylinder $L_a(r,u)$ if for every $x \in B(a,r)$ we have that $l_x(u) \cap K = [A_1(x), A_2(x)]$ with $A_i(x), i = 1,2$ being continuous for $x \in B(a,r)$ and $|A_1(x) - A_2(x)| > 0, x \in B(a,r)$.*

*Moreover, $K \subset \mathbb{R}^d$ is a **piecewise graph domain** if it can be covered by finite number of cylinders so that K is a graph domain with respect to each of them.*

If all functions $A_i(x)$ specified above are in Lip$\gamma, 0 < \gamma \leq 1$ then the piecewise graph domain K is called Lipγ.

We established the next asymptotically sharp L^p Bernstein-Markov type inequalities for cuspidal domains.

Theorem 3. *Let $K \subset \mathbb{R}^d$ be a Lip$\gamma, 0 < \gamma \leq 1$ piecewise graph domain. Then for any $q_n \in P_n^d, 1 \leq p < \infty$ and $n \in \mathbb{N}$*

$$\|Dq_n\|_{L^p(K)} \leq c(K,p)n^{\frac{2}{\gamma}}\|q_n\|_{L^p(K)}.$$

Moreover, if K if is imbedded in an affine image of the l_γ ball having one of its vertices on ∂K then there exist $g_n \in P_n^d, n \in \mathbb{N}$ such that

$$\|Dg_n\|_{L^p(K)} > c_1(K,p)n^{\frac{2}{\gamma}}\|g_n\|_{L^p(K)},\ n \in \mathbb{N}.$$

By inserting a proper weight function vanishing at the boundary of the domain one can ensure a smaller Bernstein type upper bound of order n for derivatives of polynomials of degree n. Given any $x \in K$ we denote by

$$\tau_K(x) := \inf_{y\in\partial K} |x-y|$$

the Euclidean distance from x to the boundary ∂K of the domain. As shown by the next theorem the quantity $\tau_K(x)^{\frac{1}{\gamma}-\frac{1}{2}}$ provides the proper weight function needed in order to drive a Bernstein type upper bound for Lip$\gamma, 0<\gamma\le 1$ piecewise graph domains.

Theorem 4. *Let $1 \le p < \infty$. Then given any Lip$\gamma, 0<\gamma\le 1$ piecewise graph domain $K \subset \mathbb{R}^d$ we have*

$$\|\tau_K(x)^{\frac{1}{\gamma}-\frac{1}{2}} Dq_n\|_{L^p(K)} \le cn\|q_n\|_{L^p(K)}, \quad q_n \in P_n^d. \tag{6}$$

Now we turn our attention to some recent results on L^p Markov type inequalities for homogeneous polynomials on graph domains. Theorem 2 gives a sharp upper bound in the Markov inequality for homogeneous polynomials H_n^d on $L^2(S^{d-1})$. It is remarkable that this bound is of order $\sim n$ even though for polynomials of total degree n the corresponding order is known to be of magnitude $\sim n^2$. This raises the natural question if this improvement in the rate of Markov factor for homogeneous polynomials holds on other domains, as well?

Theorem 5. *Assume that K is $C^\alpha, 1<\alpha\le 2$ star like domain in $\mathbb{R}^d$ with non degenerate outer normal. Then for any $h \in H_n^d$ and every $1\le p<\infty$ we have*

$$\|Dh\|_{L^p(K)} \le cn^{\frac{1}{\alpha}+\frac{1}{2}}\|h\|_{L^p(K)}. \tag{7}$$

Moreover, if K is Lip 1 then for every $h \in H_n^d$ and any $k \in \mathbb{N}$

$$\|Dh\|_{L^p(K)} \le cn^{\frac{3}{2}}\|h\|_{L^p(K)}.$$

In case when $\alpha = 2$ that is K is a C^2 star like domain with non degenerate outer normal Theorem 5 evidently yields an optimal order n Markov type estimate.

Corollary. *For any C^2 star like domain $K \in \mathbb{R}^d$ with non degenerate outer normal and every $h \in H_n^d$*

$$\|Dh\|_{L^p(K)} \le cn\|h\|_{L^p(K)}, \quad 1 \le p < \infty.$$

Marcinkiewicz-Zygmund type results on discretization of norms.

Historically the first discretization result was given by S.N. Bernstein in 1932 who showed that for any trigonometric polynomial t_n of degree $\le n$ and any $0 = x_0 < x_1 < ... < x_N < 2\pi = x_{N+1}$ with $\max_{0\le j\le N}(x_{j+1}-x_j) \le \frac{2\sqrt{\tau}}{n}$, $0<\tau<2$ we have

$$\max_{x\in[0,2\pi]} |t_n(x)| \le (1+\tau)\max_{0\le j\le N}|t_n(x_j)|. \tag{8}$$

The $L^q, 1<q<\infty$ analogue is due to Marcinkiewicz and Zygmund who verified in 1937 that for any univariate trigonometric polynomial t_n of degree at most n and every $1<q<\infty$

$$\int |t_n|^q \sim \frac{1}{n}\sum_{s=0}^{2n}\left|t_n\left(\frac{2\pi s}{2n+1}\right)\right|^q \tag{9}$$

We gave a refinement of the classical Marcinkiewicz-Zygmund result which is similar to Bernstein's estimate (8).

Theorem 6. *For any $-\pi = x_0 < x_1 < ... < x_m = \pi$ with $\max_{0\le j\le m-1}(x_{j+1}-x_j) < \frac{\sqrt{\tau}}{qn}$, and for every $t_n \in T_n, 1 \le q < \infty$ we have*

$$(1-\tau)\sum_{j=0}^{m-1}\frac{x_{j+1}-x_{j-1}}{2}|t_n(x_j)|^q \le \int_{-\pi}^{\pi}|t_n(x)|^q dx \le (1+\tau)\sum_{j=0}^{m-1}\frac{x_{j+1}-x_{j-1}}{2}|t_n(x_j)|^q. \tag{10}$$

This is a Marcinkiewicz-Zygmund type estimate of precision τ similar to Bernstein's uniform bound (8). In particular, choosing equidistant nodes $x_j := \frac{2\pi(j-1)}{m+1}, 1 \le j \le m+1$ with $m = \left[\frac{2\pi qn}{\sqrt{\tau}}\right]+2$ we obtain

$$\frac{1-\tau}{m}\sum_{j=1}^{m}|t_n(x_j)|^q \le \frac{1}{2\pi}\int_0^{2\pi}|t_n(x)|^q dx \le \frac{1+\tau}{m}\sum_{j=1}^{m}|t_n(x_j)|^q.$$

We also proved a new Marcinkiewicz-Zygmund type discretization result for the integral norms of **general exponential sums**. For a given $n \in \mathbb{N}, \delta, M > 0$ let us introduce the following set of *n term exponential sums* in $\mathbb{R}^d$ with exponents separated by δ and bounded by M

$$\Omega^d(n,\delta,M) := \{\sum_{1\le j\le n} c_j e^{\langle \mu_j, \mathbf{w}\rangle},\ \ c_j \in \mathbb{R},\ \mu_j, \mathbf{w} \in \mathbb{R}^d, |\mu_{j+1}-\mu_j| \ge \delta, |\mu_j| \le M\}.$$

Theorem 7. *Let $1 \le q < \infty, 0 < \delta \le 1, n \in \mathbb{N}, M > 1$. Then we can explicitly give discrete sets $Y_N = \{x_j\}_{j=1}^N \subset (a,b)$ of cardinality*

$$N \le cqn \ln^{\frac{1}{q}+1}\frac{M}{\delta}$$

so that for each exponential sum $g \in \Omega^1(n,\delta,M)$ we have

$$\|g\|^q_{L^q[a,b]} \sim \sum_{1\le j\le N-1}(x_{j+1}-x_j)|g(x_j)|^q, \tag{11}$$

where all the constants involved are absolute.

The above upper bound for the cardinality of the discrete meshes turns out to be *near optimal* in the sense that it is sharp with respect to dimension n up to the logarithmic term. The degree M and separation parameter δ of the exponential sums appearing only in the logarithmic term has a limited effect on the bound. In addition, the discrete set is **universal** in the sense that it depends only on dimension n, degree M and separation parameter δ of the exponential sums.

The above discretization result admits a generalization to the unit cube $I^d := [0,1]^d$ in $\mathbb{R}^d$.

Theorem 8. *Let $1 \le q < \infty, d, n \in \mathbb{N}, 0 < \delta < 1, M > 1$. Then there exist positive weights $a_1, ..., a_N$ and discrete point sets $Y_N = \{\boldsymbol{w}_1, ..., \boldsymbol{w}_N\} \subset I^d$ of cardinality*

$$N \le c(d,q)n^d \ln^{\frac{d}{q}+d}\frac{M}{\delta},$$

so that for every exponential sum $g \in \Omega^d(n,\delta,M)$ we have

$$\|g\|^q_{L_q(I^d)} \sim \sum_{1\le i\le N} a_i|g(\mathbf{w}_i)|^q, \tag{12}$$

where all the constants involved depend only on d and q.

Exceptional orthogonal polynomials. We computed asymptotics of the recurrence coefficients of X1-Jacobi polynomials and investigated the limit of the corresponding Christoffel functions. The proofs of the related theorems with respect to standard orthogonal polynomials are based on the three-term recurrence relation. The main new point in this respect is the fact that exceptional orthogonal polynomials possess at least five-term recurrence formulae and so a Christoffel-Darboux formula is not available in this case. These difficulties were successfully handled in a combinatorial way. In addition, estimates for the Lebesgue constants, as well as error estimates of the approximation by barycentric interpolation operators were given. Some inherited properties of exceptional Jacobi polynomials were derived and as an application it was shown that similarly to the standard case, the equilibrium measure of Julia sets of exceptional Jacobi polynomials tends to the equilibrium measure of the interval of orthogonality in weak-star sense.

Weighted inequalities for the maximal operator with respect to the discrete diffusion semigroups associated with exceptional Jacobi and Dunkl-Jacobi polynomials were given. This setup allows to extend the corresponding results obtained for discrete heat semigroup to richer class of differential-difference operators.

General translations and Bessel convolutions. We introduce general translations as solutions to Cauchy or Dirichlet problems. This point of view allows us to handle for instance the heat-diffusion semigroup as a translation. With the given examples, Kolmogorov–Riesz characterization of compact sets in certain L^p_μ spaces is derived.Pego-type characterization results are also derived. Finally, in certain model cases the equivalence of the corresponding modulus of smoothness and K-functional is pointed out.

We defined and examined nonlinear potential by Bessel convolution with Bessel kernel. We investigated removable sets with respect to Laplace-Bessel equation.By studying the maximal and fractional maximal measure, a Wolff type inequality was proved. Finally the relation of $B-p$ capacity and B-Lipschitz mapping, and the $B-p$ capacity and weighted Hausdorff measure and the $B-p$ capacity of Cantor sets has been examined.