PROJECT CLOSING REPORT – PD128374

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Our work was devoted to the study of *(a)* operators acting on anti-dual pairs and *(b)* preserver transformations on structures of operators [absolute continuity and singularity preserving maps] and on metric spaces of measures [distance preserving maps on Wasserstein spaces]. Our results obtained in the course of the project were formulated in 11 papers, some of them are not yet published. In what follows, we briefly summarize the obtained results.

1. Operators on anti-dual pairs

Our main aim in [12] was to generalize the classical concept of positive operator, and to develop a general extension theory, which overcomes not only the lack of a Hilbert space structure, but also the lack of a normable topology. The concept of anti-duality carries an adequate structure to define positivity in a natural way, and is still general enough to cover numerous important areas where the Hilbert space theory cannot be applied. Our running example – illustrating the applicability of the general setting to spaces bearing poor geometrical features – came from noncommutative integration theory. Namely, representable extension of linear functionals of involutive algebras was governed by their induced operators. The main theorem, to which the vast majority of the results is built, gives a complete and constructive characterization of those operators that admit a continuous positive extension to the whole space. Various properties such as commutation, or minimality and maximality of special extensions was studied in detail.

In [5] we developed an approach to obtain self-adjoint extensions of symmetric operators acting on anti-dual pairs, and we showed how hermitian extensions of linear functionals of involutive algebras can be governed by means of their induced operators. As an operator theoretic application, we provided a direct generalization of Parrott's theorem on contractive completion of 2 by 2 block operator-valued matrices. To exhibit the applicability in noncommutative integration, we characterized hermitian extendibility of symmetric functionals defined on a left ideal of a C^*-algebra.

The goal of [13] was to develop the theory of Schur complementation in the context of operators acting on anti-dual pairs. As a byproduct, we obtained a natural generalization of the parallel sum and parallel difference, as well as the Lebesgue-type decomposition. To demonstrate how this operator approach works in application, we derive the corresponding results for operators acting on rigged Hilbert spaces, and for representable functionals of *-algebras.

2. Maps preserving absolute continuity and singularity

In [3] we considered the cone of all positive, bounded operators acting on an infinite dimensional, complex Hilbert space, and examined bijective maps that preserve absolute continuity in both directions. It turned out that these maps are exactly those that preserve singularity in both directions. Moreover, in some weak sense, such maps are always induced by bounded, invertible, linear- or conjugate linear operators of the underlying Hilbert space. Our main result (see below) gives a possible generalization of a recent theorem of Molnár which characterizes maps on the positive cone that preserve the Lebesgue decomposition of operators.

3. Isometries of Wasserstein spaces

The vast majority of the research conducted during the project belongs to this topic. Our main aim was to describe the structure of *isometries* (i.e., distance preserving bijections) and *isometric embeddings* (i.e., not necessarily surjective distance preserving maps) of various metric spaces of measures.

The central object of our investigations is the so-called Wasserstein space which is roughly speaking the space of sufficiently concentrated probability measures endowed with a metric which is calculated by means of optimal transport. The precise definition is the following: let (X, ϱ) be a complete and separable metric space, and denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X. For two measures $\mu, \nu \in \mathcal{P}(X)$ a Borel probability measure π on $X \times X$ is said to be a *coupling* of (or *transport plan*) if the marginals of π are μ and ν, that is,

$$\pi\left(A \times X\right) = \mu(A) \qquad \text{and} \qquad \pi\left(X \times B\right) = \nu(B)$$

for all Borel sets $A, B \subseteq X$. The set of all couplings is denoted by $\Pi(\mu, \nu)$. For any parameter value $0 < p < \infty$ one can define the *p-Wasserstein space* $\mathcal{W}_p(X)$ as the set of all $\mu \in \mathcal{P}(X)$ that satisfy $\int_X \varrho(x, \hat{x})^p \, \mathrm{d}\mu(x) < \infty$ for some (hence all) $\hat{x} \in X$, endowed with the *p-Wasserstein distance*

$$d_{\mathcal{W}_p}\left(\mu, \nu\right) := \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} \varrho(x, y)^p \, \mathrm{d}\pi(x, y) \right)^{\min\left\{\frac{1}{p}, 1\right\}}.$$

This distance measures the minimal effort required to morph one probability mass into another, when the cost of transporting mass is the p-th power of the distance.

A very important feature of Wasserstein spaces is that if $p \geq 1$ then X can be embedded isometrically into $\mathcal{W}_p(X)$ via Dirac masses, but the image of this embedding is a very small set. Similarly, the isometry group of X can be embedded into the isometry group of $\mathcal{W}_p(X)$ by means of push-forwards. Those isometries of $\mathcal{W}_p(X)$ that are push-forwards of some isometry of X are called *trivial* isometries. Recall that the *push-forward map* $g_\#\colon \mathcal{P}(X) \to \mathcal{P}(X)$ induced by a measurable function $g\colon X \to X$ is defined by $\big(g_\#(\mu)\big)(A) = \mu(g^{-1}[A])$ for all $A \subseteq X$ and $\mu \in \mathcal{P}(X)$, where $g^{-1}[A] = \{x \in X : \, g(x) \in A\}$.

The main question of our research is the following: is it possible that although $\mathcal{W}_p(X)$ is a much larger metric space than X, but still, their isometry groups are isomorphic to each other? In that case we will call $\mathcal{W}_p(X)$ *isometrically rigid*.

Since the metric itself depends heavily on the value of p and the geometry of the underlying structure, different inputs lead to completely different problems that are independent of each other. Even if (X, ϱ) is fixed, the cases $0 < p < 1$, $p = 1$, $p = 2$, and $p > 1 (p \neq 2)$ must be handled separately because of the structure of optimal transport plans are essentially different.

First we considered the case when (X, ϱ) is a countable set endowed with the trivial metric. The main achievement in [2] is that we were able to describe not only the structure of isometries but the structure of isometric embeddings, which is usually incomparably more difficult. In this case, we found that each isometry of $\mathcal{W}_p(X)$ is induced by a bijection of X, while isometric embeddings of $\mathcal{W}_p(X)$ can be described by a special kind of $X \times (0, 1]$-indexed family of nonnegative finite measures.

As a generalization, we considered Wasserstein spaces on graph metric spaces in [11]. We say that a countable metric space (X, ϱ) is a graph metric space if there exists a connected graph $G := G(X, E)$ with vertex set X and edge set E, such that the edge-length of the shortest path between any two vertices $a, b \in X$ equals to $\varrho(a, b)$. Note that the setting of [2] corresponds to the case of the complete graph. The class of graph metric spaces contains many important metric spaces, just to mention a few: the set of natural numbers and the set of integers with the usual $| \cdot |$-distance; d-dimensional lattices endowed with the l^1-metric or the l^∞-metric and finite strings with the Hamming distance (the classical 1-Wasserstein distance with respect to the Hamming metric is called Ornstein's distance). In [11] we proved that the Wasserstein space $\mathcal{W}_p(X)$ is isometrically rigid for all $p \geq 1$ whenever X is a countable graph metric space. As a consequence, we obtained that for every countable group H there exists a Wasserstein space whose isometry group is isomorphic to H.

Due to the peculiarities of the discrete structure, the results of [2] and [11] did not show that the structure of the isometries of $\mathcal{W}_p(X)$ really depends on the value of p. However, the differences will be clearly seen when we take a look at the case of the real line, that is when $(X, \varrho) = (\mathbb{R}, | \cdot |)$. While investigating some geometric properties of the quadratic Wasserstein space $\mathcal{W}_2(\mathbb{R}^n)$, Kloeckner pointed out in 2010 that there is an important difference between the cases of $X = \mathbb{R}$ and $X = \mathbb{R}^n$ $(n \geq 2)$. Namely $\mathcal{W}_2(\mathbb{R})$ contains a one-parameter subgroup of non-trivial isometries. These isometries are called exotic, as they behave quite wildly: although they preserve the set of Dirac masses, they distort the shape of measures in general. In [10] we showed that the exotic isometry flow extends into a unitary group on $L^2((0, 1))$, and we computed its antisymmetric generator on a dense domain. As an application, we showed that every exotic isometry (and thus all isometry) maps the set of all absolutely continuous measures belonging to $\mathcal{W}_2(\mathbb{R})$ onto itself. This paper is in preparation and will be submitted for publication soon.

Back to the isometry group of $\mathcal{W}_p(\mathbb{R})$ and the role of the underlying space and the value of p: motivated by Kloeckner's striking result and the fact the one-dimensional case in optimal transportation looks very special, in our Trans. Amer. Math. Soc. paper [4] we described the isometry group of $\mathcal{W}_p(\mathbb{R})$ for all $p \geq 1$. It turned out that $\mathcal{W}_p(\mathbb{R})$ is isometrically rigid if and only if $p \neq 2$. That is, the exotic isometry flow disappears for all $p \neq 2$. Surprisingly, if we replace $\mathbb{R}$ by $[0,1]$ and we consider the Wasserstein space $\mathcal{W}_p([0,1])$, then find that $\mathcal{W}_1([0,1])$ admits exotic isometries, and every other $\mathcal{W}_p([0,1])$ is isometrically rigid. It is important to mention that exotic isometries of $\mathcal{W}_1([0,1])$ can split masses, that is, the image of a Dirac measure is not necessarily a Dirac measure. We wrote about this phenomenon and in general the role of Dirac masses in [1].

Another important result in [4] is about isometric embeddings: in the $p > 1$ $p \neq 2$ case we proved that any isometric embedding of $\mathcal{W}_p(\mathbb{R})$ can be written as a composition of a trivial isometry, and a special kind of translation which is defined by means of the inverse distribution functions of the measures in question.

In [8] we presented an even broader extension of Kloeckner's result on the isometry group of $\mathcal{W}_2(\mathbb{R}^n)$. Namely, we were able to handle the case of arbitrary p, and to drop the assumption of finite-dimensionality. It turned out that the case $p = 2$ is again exceptional: for any separable real Hilbert space E, the Wasserstein space $\mathcal{W}_p(E)$ admits nontrivial isometries if and only if $p = 2$.

Up to this point, we have never mentioned the case $0 < p < 1$. One of the most important results in [8] is that we proved that if $0 < p < 1$ then the Wasserstein space $\mathcal{W}_p(X)$ is isometrically rigid, regardless of what the underlying metric space X is. From the theoretical point of view, this case is interesting because the transport plans have a rather different structure. From the economic point of view, this setting seems to be the most natural one when moving a mass has a cost which is proportionally less if the distance increases.

Furthermore, in [8] we answered a question raised by Kloeckner affirmatively by showing that for all $p > 1$ there exists a Polish space X such that the p-Wasserstein space $\mathcal{W}_p(X)$ possesses mass splitting isometries.

In [9] we investigated two important special cases: when the underlying space (X, ϱ) is the d-dimensional torus, and when (X, ϱ) is the d-dimensional sphere. Although these sets are very nice subsets of $\mathbb{R}^d$, we cannot expect that the techniques and results obtained in the euclidean case are applicable here. We recall the case of $X = [0,1]$and $X = \mathbb{R}$. While $\mathcal{W}_1(\mathbb{R})$ is isometrically rigid, that is, the isometry group of $\mathcal{W}_1(\mathbb{R})$ is isomorphic to the isometry group of $\mathbb{R}$, the isometry group of $\mathcal{W}([0,1])$ is the Klein group. Similarly, $\mathcal{W}_2(\mathbb{R})$ possesses a flow of wild behaving isometries, while $\mathcal{W}_2([0,1])$ is rigid. In [9] we showed that all $\mathcal{W}_p(\mathbb{S}^d)$ and $\mathcal{W}_p(\mathbb{T}^d)$ spaces are isometrically rigid. The key idea in this paper is that in some cases a measure can be recovered from its potential function. Since it seems that this method works in more general circumstances as well, we decided to delay the publication until we find the most general form.

4. OTHER ACTIVITIES

I gave several scientific talks during the course of the project, I list here only the most important ones.

- 19. 12. 2018. - Positive operators on anti-dual pairs, 2018 China-Korea International Conference on Matrix Theory with Applications, Shanghai University, China
- 29. 01. 2019. - On the isometry group of metric spaces of probability measures, Researches on isometries as preserver problems and related topics, Kyoto RIMS, Japan
- 12. 07. 2019. - Positive operators on anti-dual pairs: Parallel addition, ILAS 2019 - Invited Mini-Symposium: Matrix Analysis, Rio de Janeiro, Brasil
- 22. 11. 2019. - Distance preserving maps on spaces of probability measures, 8th OPSFOTA Meeting and Operator Theory Workshop, University of Reading, UK
- 12. 02. 2020. - Isometries of Wasserstein spaces, Mathematics and CS Seminar, Institute of Science and Technology Austria, Klosterneuburg, Austria
- 17. 02. 2020. - Isometries of Wasserstein spaces, Rings, Modules and Matrices Seminar, Lomonosov Moscow State University Moscow, Russia
- 17. 11. 2020. - Isometries of Wasserstein spaces, Analysis Seminar, University of Wisconsin-Madison, Madison, USA
- 23. 06. 2021. - Maps preserving absolute continuity of positive operators, 8th European Congress of Mathematics, MS: Recent Developments on Preservers, Portoroz, Slovenia

I not only participated at conferences but I was also involved in the organization of workshops and seminars.

With György Pál Gehér and Dániel Virosztek we started an online seminar series called "Preserver Webinar" (later we changed the topic slightly, the new name is Functional Analysis and Operator Theory Webinar – with an emphasis on preserver problems and quantum information). We provide here a short list of important invited speakers: Chi-Kwong Li (The College of William and Mary, Virginia), Michiya Mori (University of Tokyo, Japan), Lajos Molnár (University of Szeged, Hungary), Peter Semrl (University of Ljubljana, Slovenia), Bas Lemmens (University of Kent, United Kingdom), Javad Mashreghi (Laval University, Canada), Andreas Winter (Universitat Autònoma de Barcelona, Spain), Karol Życzkowski (Jagiellonian University and Polish Academy of Sciences, Poland), Apoorva Khare (Indian Institute of Science, India), Benoît R. Kloeckner (L'Université Paris-Est Créteil, France)

Furthermore, with Lajos Molnár we organized an international one weekend workshop for preserver problems in 2019 in Szeged.

Finally, I mention that I also wrote three articles [6, 7, 14] in Hungarian for the "What is..." section of the electronic mathematical magazine of the János Bolyai Mathematical Society. The aim of these papers was to introduce and popularize the subject to the general audience.

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