During the first year of the research project, two articles have been published in Q1 journals. As a result of collaborations, two additional working papers have been published on topics related to the proposed plan of the research project PD-128348. Note that this project has been stopped as of September 2019 and this request was approved by NKFIH.

A paper titled "A note on the relationship between the core and stable sets in three-sided markets" is published at Mathematical Social Sciences (see Atay and Núñez [1]). This paper answers questions posed for the second year of this project. In this paper, the purpose is to study the existence of (von Neumann-Morgenstern) stable sets and its relationship with the core. The fact that the core may be empty makes the notion of stable set more appealing as a solution concept for multi-sided assignment games. As there have been no results focusing on stable sets, this paper sheds light on stability for multi-sided assignment games.

We analyze the extent to which two known results of the relationship between the core and the stable sets for two-sided assignment games can be extended to three-sided assignment games. First, we study whether the dominant diagonal property is a necessary and sufficient condition for the core to be a stable set, as in two-sided markets. Second, we analyze whether the union of the extended cores of all mu-compatible subgames is a stable set of the three-sided assignment game, as in the two-sided markets.
To study the first question, we generalize the notion of dominant diagonal to the three-sided case and prove that it is a necessary condition for the core of this game to be a stable set. Moreover, we show that, for three-sided markets with only two agents on each side, the dominant diagonal property is a sufficient condition for the core of this game to be a stable set. Although we show that the dominant diagonal property is a necessary condition for three-sided assignment markets and it characterizes the core stability when there are two agents in each side, it remains open as to whether it is also sufficient for arbitrary three-sided assignment games. I plan to return on this open question in the second and the third year of the project.

In the second part of this paper, we focus on the existence of stable sets for three-sided assignment games. To do so, we first extend the notion of mu-compatible subgames defined for the two-sided case. Following this notion, for a given three-sided assignment market, we consider the set V consisting of the extended cores of all mu-compatible subgames. We study the stability properties of this set. We show that this set may fail to satisfy external stability. Hence, in general, it is not a stable set. Although V is not a stable set, we show that this set does have some appealing stability properties for three-sided assignment games. In a three-sided assignment game, it seems natural to restrict the set of feasible outcomes to those imputations that are compatible with some optimal matching. We prove that, when we consider the aforementioned outcomes as feasible, the set V is precisely the set of undominated outcomes, that is, the "core" with respect to the set of imputations that are compatible with some optimal matching.

The second paper titled "Multi-sided assignment games on m-partite graphs" is published at Annals of Operations Research. This paper deals with the third year of the research project. The main aim is to study how to make allocations in a "fair" and "optimal" way in multi-sided matching markets with payments (e. g. production lines).

In this paper, see Atay and Núñez [2], we consider generalized multi-sided assignment games where cooperation is restricted by an underlying network structure. When there exist more than two sides in the market, the core may be empty. Hence, most relevant properties cannot be extended to the multi-sided markets. To overcome these, several authors have proposed specific

classes of multi-sided assignment games where the core is non-empty and may preserve some appealing structural properties.

In this paper, we consider a graph on the set of sectors which indicates which sectors in the market are linked. This graph induces an m-partite graph on the set of agents: if two sectors are connected, each mixed-pair of agents of these sectors is connected in the m-partite graph. The weights on the graph define an underlying two-sided assignment market for each pair of connected sectors and by additivity give rise to the value of basic coalitions. Then, we define a coalitional game, the corresponding multi-sided assignment game.

In this new setting, multi-sided assignment games on an m-partite graph, we introduce sufficient conditions on the weights that guarantee the non-emptiness of the core. When we impose that the underlying structure of the graph that connects sectors is cycle-free, we guarantee non-emptiness of the core regardless the system of weights. Moreover, we characterize the core of a multi-sided assignment game on an m-partite graph where the quotient graph on the set of sectors is cycle-free. We show that the core is fully described by the "composition" of the cores of all underlying two-sided markets when the quotient graph on the set of sectors is cycle-free. As a consequence, we study properties of the core of these multi-sided assignment games by means of the cores of their underlying two-sided games. Following our characterization, we prove that: *(i)* there exist optimal core allocations for each sector, *(ii)* extreme core allocations are obtained from the core extreme-points of two-sided markets. Furthermore, we provide counterexamples to show that these results do not hold for arbitrary multi-sided assignment games. Finally, we analyze single-valued solutions with a fairness property when the quotient graph on the set of sectors is cycle-free. We show that the nucleolus of a multi-sided assignment market on an m-partite graph need not coincide with the composition of nucleoli of underlying two-sided markets. Also, we provide an example where the fair-division point is not a core element, whereas the composition of fair-division points of each underlying two-sided game lies in the core. Hence, we guarantee the existence of a stable allocation with a fairness property.

During the period of September 2018-September 2019, the results were presented in invited seminars at Aix-Marseille School of Economics, University of Saint-Louis, UCLouvain, and University of Barcelona. Also, I gave talks at the international conferences: Conference on Economic Design 2019 and SING 15-European Meeting on Game Theory.

I have also worked with Ana Mauleon and Vincent Vannetelbosch from UCLouvain on a topic related to the research plan of the project PD-128348. In this project, we focus on matching problems under preferences. We consider matching markets without payments where there exists a set of agents each endowed with preferences over all agents. Each agent is interested in forming at most one partnership. This model is a generalization of the stable marriage problem which can be seen as the counterpart of the two-sided assignment game. A matching is said to be stable if there is no agent who prefers being unmatched to her prescribed partner or no pair of agents prefer being matched to each other to their current partners. It is known that stable matchings always exist for the marriage problem, whereas their existence is not guaranteed for the roommate problem. In this project, instead of restricting the analysis to roommate problems with stable matchings, we adopt a weaker notion of stability for solving the roommate problem: the bargaining set. First, we guarantee the existence of weakly stable matchings which are based on a weaker stability notion. Second, we show that weak stability is not sufficient for a matching to be in the bargaining set. Moreover, when the set of stable matchings (the core) is non-empty, in general, it is a subset of the set of weakly stable matchings. Third, we prove that the bargaining set is always non-empty. Finally, we show that the bargaining set coincides with the

set of weakly stable and weakly efficient matchings in the roommate problem. We propose a set-wise solution concept (the bargaining set) based on a weaker stability notion which is always non-empty and contains stable matchings whenever they exist. Hence, the bargaining set can be used as a solution concept for roommate problems which has important applications such as coalition formation and kidney exchange problem. A working paper on this project has been published (see Atay et al. [3]).

During the first year of the project, another working paper has been published (see Atay et al. [4]). We analyze the core of coalitional games for unit time open shop scheduling problems since core elements would provide a stable reallocation of the joint cost savings. By supposing an initial schedule and associating each job (consisting of a number of operations) to a different player, we construct a cooperative TU-game associated with any open shop scheduling problem. We assign to each coalition the maximal cost savings it can obtain through admissible rearrangements of jobs' operations. We provide a stable allocation of the total cost savings obtained by cooperation for unit open shop scheduling problems. Finally, we study to what extend such allocation is still stable if we relax the definition of admissible rearrangements for a coalition.

Note that these two working papers are published in 2021. Since I am not affiliated with the Insitute of Economics and stopped the project by September 2021, the published versions of those projects are not added to the report. Nevertheless, I would like to point out that as both projects have started as a part of PD- 128348, I gratefully acknowledged the support in both articles.

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