# Project closing report (final report): 2018-2022

**Project no.: 127926, K-18**

**Project Leader: Alexandru Kristály , Óbuda University, Budapest**

FUNCTIONAL INEQUALITIES AND ELLIPTIC PDES: THE INFLUENCE OF CURVATURE
Funkcionál-egyenlőtlenségek és elliptikus parciális differenciálegyenletek: a görbület hatása

**Abstract**

In this document we report the scientific achievements obtained in the period 2018-2022, supported by the National Research, Development and Innovation Fund of Hungary, financed under the K_18 funding scheme, Project no. 127926, K-18.

**MEMBERS OF THE RESEARCH TEAM:**

- **Project Leader:** dr. Alexandru Kristály, Full Professor;
- **Member:** dr. Csaba Farkas, Postdoctoral Student;
- **Member:** Ágnes Mester, PhD Student.

## Contents

# 1 General presentation

The primordial aim of the project was to establish new functional inequalities on curved spaces and to show their applicability in the theory of elliptic partial differential equations (shortly, elliptic PDEs). Our main objectives were:

1) *Prove sharp functional inequalities on nonpositively curved spaces.*

2) *Describe topological properties of nonnegatively curved spaces supporting sharp functional inequalities.*

3) *Guarantee existence of solutions for elliptic PDEs on Riemannian manifolds.*

4) *Study of elliptic PDEs on non-compact Finsler manifolds.*

During the last 4 years we proved fine quantitative and qualitative results where the curvature played a central role in various non-Euclidean settings. In our studies, we used a diversity of arguments from the calculus of variations, Riemann/Finsler geometry, measure and group theory.

In the sequel, we present shortly our scientific achievements (published/accepted papers), conferences/talks and mobilities within the project and some final notes.

# 2 Published/accepted papers

We are happy that we had a very fruitful period of research (2018-2022), when we published in top journals, as in *Advances in Mathematics, Proceedings of the London Mathematical Society, Calc. Var. & PDEs, Transactions of the American Mathematical Society, Journal of Differential Equations, Mathematische Zeitschrift.* Quantitatively, we published

- **9** of D1-type papers[1];
- **5** of Q1-type (not D1) papers;
- **1** of Q2-type paper;
- **1** Springer-monograph;
- **2** conference papers.

In the sequel, we list the published/accepted papers together with some short descriptions, all of them being supported by the present project.

1. **Kristály A.**, Szakál A., *Interpolation between Brezis-Vázquez and Poincaré inequalities on nonnegatively curved spaces: sharpness and rigidities*, **J. DIFFERENTIAL EQUATIONS**, 266 (2019), no. 10, 6621–6646. [SCIMAGO classification: **D1**] Repository no.: 101225; arXiv: 1811.04200.

   This paper is devoted to investigate an interpolation inequality between the Brezis-Vázquez and Poincaré inequalities (shortly, BPV inequality) in $CD(K,n)$ spaces á la Lott-Sturm-Villani, with $K \geq 0$ and $n > 1$. Our main result proves that if a $CD(K,n)$ metric measure space supports the BPV inequality and a lower $n$-density property at some point, then the measure of every metric ball coincides with the volume of its Euclidean counterpart. Further rigidity results are

[1] Journals ranking in MTMT database. D1: the first 10%, Q1: the first 25%, Q2: 25%-50%.

also derived on smooth Finsler/Riemannian manifolds. For instance, a Berwald/Riemannian manifold of nonnegative Ricci curvature supports the BPV inequality if and only it is isometric to a Minkowski/Euclidean space; in the latter case we fully characterize the extremals in the BPV inequality. Our arguments explore fine properties of Bessel functions, the Bishop-Gromov comparison principle in $CD(K,n)$ spaces, and anisotropic symmetrization on Minkowski spaces. We also characterize the existence of nonzero solutions for a quasilinear PDE involving the Finsler-Laplace operator and a Hardy-type singularity on Minkowski spaces where the sharp BPV inequality plays a crucial role. Paper in full conformity with Problems 2 and 4.

2. **Kristály A., Mester Á.**: *A bipolar Hardy inequality on Finsler manifolds*, IEEE 13TH INTERNATIONAL SYMPOSIUM ON APPLIED COMPUTATIONAL INTELLIGENCE AND INFORMATIC, pp. 308-312, 2019. Repository no.: 101251; arXiv:2010.06316.

   We establish a bipolar Hardy inequality on complete, not necessarily reversible Finsler manifolds. We show that our result strongly depends on the geometry of the Finsler structure, namely on the reversibility constant $r_F$ and the uniformity constant $l_F$. Our result represents a Finslerian counterpart of some well-known Euclidean multipolar Hardy inequality in the sense of Cazacu C. and Zuazua E. Paper in full conformity with Problems 1 and 4.

3. **Kristály A.**, *Fundamental tones of clamped plates in nonpositively curved spaces*, **ADV. MATH.** 367 (2020), 107113, p. 39. [SCIMAGO classification: **D1**] Repository no.: 114919; arXiv:1909.02350.

   We study Lord Rayleigh's problem for clamped plates on Cartan-Hadamard manifolds $(M,g)$. In 2- and 3-dimensions we prove sharp isoperimetric inequalities for sufficiently small clamped plates, i.e. the fundamental tone of any domain in $(M,g)$ of volume $v>0$ is not less than the corresponding fundamental tone of a geodesic ball of the same volume $v$ in the space of constant negative curvature provided that $v$ is sufficiently small. The sharpness of our results requires the strong-Cartan-Hadamard conjecture and peculiar properties of the Gaussian hypergeometric function, both valid only in dimensions 2 and 3. We also give necessary and sufficient conditions for the existence of a nontrivial solution to an elliptic PDE involving the biharmonic Laplace-Beltrami operator. Paper in full conformity with Problems 1 and 3.

4. Huang L., **Kristály A.**, Zhao W., *Sharp uncertainty principles on general Finsler manifolds*, **TRANS. AMER. MATH. SOC.**, 373 (2020), no. 11, 8127–8161. [SCIMAGO classification: **D1**] Repository no.: 101255; arXiv: 1811.08697.

   The paper is devoted to sharp uncertainty principles on forward complete Finsler manifolds endowed with an arbitrary measure. Under mild assumptions, the existence of extremals corresponding to the sharp constants in the Heisenberg-Pauli-Weyl and Caffarelli-Kohn-Nirenberg inequalities fully characterizes the nature of the Finsler manifold in terms of three non-Riemannian quantities. Paper in full conformity with Problems 1 and 2.

5. **Kristály A.**, Mezei I. I., Szilák K., *Differential inclusions involving oscillatory terms*, **NONLINEAR ANALYSIS**, 197 (2020), 111834, p. 21. [SCIMAGO classification: **D1**] Repository no.: 114938; arXiv: 2002.12678.

   Motivated by mechanical problems where external forces are non-smooth, we consider a differential inclusion problem on a bounded open domain. We provide a complete picture on the number of solutions of the studied problem whenever the terms oscillate near the origin/infinity and the other term is a generic perturbation of order $p>0$. Paper in conformity with Problem 3.

6. Faraci F., **Farkas C.**, *On a critical Kirchhoff-type problem*, **NONLINEAR ANALYSIS**, 192 (2020), 111679, 14 pp. [SCIMAGO classification: **D1**] Repository no.: 101257; arXiv: 1907.05581.

   In the present paper the authors study a Kirchhoff type problem involving a critical term. They give sufficient conditions for the sequentially weakly lower semicontinuity and the Palais-Smale property of the energy functional associated to the problem. Paper related to Problem 3.

7. Balogh Z., Gutiérrez E. C., **Kristály A.**, *Sobolev inequalities with jointly concave weights on convex cones*, **PROC. LOND. MATH. SOC.**, 122 (2021), no. 4, 537–568. [SCIMAGO classification: **D1**] Repository no.: 114913; arXiv no.: 2003.12157.

   Using optimal mass transport, we prove new weighted Sobolev inequalities, involving two homogeneous weights verifying a general concavity-type structural condition. This study is strongly motivated by (axially invariant) Sobolev inequalities on the sub-Riemannian Heisenberg groups. The obtained Sobolev constant is given by an explicit formula. Under mild regularity assumptions on the weights, we also prove that the Sobolev constant is optimal if and only if the two weights are equal. Existence of solutions for elliptic partial differential equations is also provided. Paper in full conformity with Problem 3.

8. **Farkas Cs., Kristály Á., Mester A.**, *Compact Sobolev embeddings on non-compact manifolds via orbit expansions of isometry groups.* **CALC. VAR. PARTIAL DIFFERENTIAL EQUATIONS** 60 (2021), no. 4, Paper No. 128, 31 pp. [SCIMAGO classification: **D1**] Repository no.: 115927; arXiv no.: 2010.06282.

   Given a complete non-compact Riemannian manifold $(M, g)$ with certain curvature restrictions, we introduce an expansion condition concerning a group of isometries $G$ of $(M, g)$ that characterizes the coerciveness of $G$ in the sense of Skrzypczak and Tintarev (*Arch. Math.*, 2013). Furthermore, under these conditions, compact Sobolev-type embeddings à la Berestycki-Lions are proved for the full range of admissible parameters (Sobolev, Moser-Trudinger and Morrey). We also consider the case of non-compact Randers-type Finsler manifolds; sharpness is shown by means of the Funk model. Paper in full conformity with Problems 2-4.

9. **Farkas Cs.**, Winkert P., *An existence result for singular Finsler double phase problems*, **J. DIFFERENTIAL EQUATIONS**, 286 (2021), 455-473. [SCIMAGO classification: **D1**] Repository no.: 132369; arXiv no.: 2011.03774.

   In this paper, the authors study a class of singular double phase problems defined on Minkowski spaces in terms of Finsler manifolds and with right-hand sides that allow a certain type of critical growth for such problems. Under very general assumptions and based on variational tools, the existence of at least one nontrivial weak solution for such a problem is guaranteed. This is the first work dealing with a Finsler double phase operator even in the non-singular case. Paper in conformity with Problems 3 and 4.

10. Kajántó S., **Kristály A.**, *Unexpected behaviour of flag and S-curvatures on the interpolated Poincaré metric.* **JOURNAL OF GEOMETRIC ANALYSIS** 31 (2021), no. 10, 10246–10262. [SCIMAGO classification: **Q1**] Repository no.: 132370.

    We endow the 2-dimensional disc D (with radius 2) with a parameter-dependent Poincaré-type Randers metric that linearly interpolates between the usual Riemannian Poincaré disc model ($\lambda = 0$, having constant sectional curvature $-1$ and zero $S$-curvature) and the Finsler–Poincaré metric ($\lambda = 1$, having constant flag curvature $-1/4$ and constant $S$-curvature with isotropic factor $1/2$), respectively. Contrary to our intuition, we show that when $\lambda \to 1$, both the flag and normalized

$S$-curvatures of the metric blow up close to the boundary of D for some particular choices of the flagpoles. Paper in conformity with Problem 4.

11. **Farkas Cs.**, Fiscella A., Winkert P., *Singular Finsler double phase problems with nonlinear boundary condition*, **ADVANCED NONLINEAR STUDIES**, 21 (2021), no. 4, 809–825, 2021. [SCIMAGO classification: **Q1**] Repository no.: 151676; arXiv: 2102.05467.

    In this paper the authors study a singular Finsler double phase problem with a nonlinear boundary condition and perturbations that have a type of critical growth, even on the boundary. Based on variational methods in combination with truncation techniques they prove the existence of at least one weak solution for this problem under very general assumptions. Even in the case when the Finsler manifold reduces to the Euclidean norm, their work is the first one dealing with a singular double phase problem and nonlinear boundary condition. Paper in conformity with Problem 4.

12. **Mester Á.**, Peter I.R., Varga C., *Sufficient criteria for obtaining Hardy inequalities on Finsler manifolds*, **MEDITERR. J. MATH.** 18 (2021), no. 2, Paper No. 76, 22 pp. [SCIMAGO classification: **Q2**] Repository no.: 115925; arXiv: 2010.06289.

    The authors establish Hardy inequalities involving a weight function on complete, not necessarily reversible Finsler manifolds. They prove that the superharmonicity of the weight function provides a sufficient condition to obtain Hardy inequalities. Paper in conformity with Problem 4.

13. **Mester A., Kristály A.**, *Three isometrically equivalent models of the Finsler-Poincaré disk*, IEEE 15TH INTERNATIONAL SYMPOSIUM ON APPLIED COMPUTATIONAL INTELLIGENCE AND INFORMATICS (SACI 2021) Pages: 403-407. Published: 2021. Repository no.: 132367; arXiv: 2204.03052.

    We present the isometry between the 2-dimensional Funk model and the Finsler-Poincaré disk. Then, we introduce the Finslerian Poincare upper half plane model, which turns out to be also isometrically equivalent to the previous models. As application, we state the gapless character of the first eigenvalue for the aforementioned three spaces. Paper in conformity with Problem 4.

14. **Monograph:** Costea N., **Kristály A.**, Varga Cs., *Variational and Monotonicity Methods in Nonsmooth Analysis*, **Springer/Birkhauser**, Frontiers in Mathematics, 2021. ISBN 978-3-030-81670-4. Repository no.: 132372. Link: https://link.springer.com/book/10.1007

    The present monograph provides a comprehensive presentation of a wide variety of nonsmooth problems arising in nonlinear analysis, game theory, engineering, mathematical physics, and contact mechanics. The subject matter of the monograph had its genesis in the early works of F. Clarke, who paved the way for the modern development of nonsmooth analysis. Our initial aim is to cover various topics in nonsmooth analysis, based mainly on variational methods and topological arguments. The present work includes recent achievements, mostly obtained by the authors during the last 10 years (four main parts, divided into 13 chapters), putting them into the context of the existing literature. Parts of the monograph in conformity with Problems 3 and 4.

15. **Kristály A.**, *New features of the first eigenvalue on negatively curved spaces*, **ADV. CALC. VAR.**, 15 (2022), no. 3, 475–495. [SCIMAGO classification: **Q1**] Repository no.: 114935; arXiv: 1810.06487.

    The paper is devoted to the study of fine properties of the first eigenvalue on negatively curved spaces. We first provide asymptotically sharp harmonic-type expansions of the first eigenvalue for large geodesic balls in the hyperbolic space. We then give a synthetic proof of Cheng's sharp eigenvalue comparison theorem in metric measure spaces satisfying a Bishop-Gromov-type volume monotonicity hypothesis. We also provide an example of simply connected, non-compact Finsler manifold with constant negative flag curvature whose first eigenvalue is zero; this result is in a sharp contrast with its celebrated Riemannian counterpart due to McKean (*J. Diff. Geom.* 1970). Paper in full conformity with Problems 1 and 4.

16. **Kristály A.**, Shen Z., Yuan L., Zhao W., *Nonlinear spectrums of Finsler manifolds*, **MATHEMATISCHE ZEITSCHRIFT**, 300 (2022), no. 1, 81–123. [SCIMAGO classification: **Q1**] Repository no.: 132358; arXiv no.: 1907.01182.

    In this paper we investigate the spectral problem in Finsler geometry. Due to the nonlinearity of the Finsler–Laplacian operator, we introduce faithful dimension pairs by means of which the spectrum of a compact reversible Finsler metric measure manifold is defined. Various upper and lower bounds of such eigenvalues are provided in the spirit of Cheng, Buser and Gromov, which extend in several aspects the results of Hassannezhad, Kokarev and Polterovich. Moreover, we construct several faithful dimension pairs based on Lusternik–Schnirelmann category, Krasnoselskii genus and essential dimension, respectively; however, we also show that the Lebesgue covering dimension pair is not faithful. As an application, we show that the Bakry–Émery spectrum of a closed weighted Riemannian manifold can be characterized by the faithful Lusternik–Schnirelmann dimension pair. Paper in conformity with Problems 1, 2 and 4.

17. **Farkas Cs.**, Fiscella A., Winkert P., *On a class of critical double phase problems*, **J. MATH. ANAL. APPL**. 515 (2022). Paper no.: 126420. [SCIMAGO classification: **Q1**] Repository no.: 151680; arXiv: 2107.12835.

    In this paper the authors study a class of double phase problems involving critical growth, namely

$$\begin{aligned} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\big) &= \lambda|u|^{\vartheta-2}u+|u|^{p^*-2}u \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

    where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $1<\vartheta<p<q<N$, $\frac{q}{p}<1+\frac{1}{N}$ and $\mu(\cdot)$ is a nonnegative Lipschitz continuous weight function. The operator involved is the so-called double phase operator, which reduces to the $p$-Laplacian or the $(p,q)$-Laplacian when $\mu\equiv 0$ or $\inf\mu>0$, respectively. Based on variational and topological tools such as truncation arguments and genus theory, they show the existence of $\lambda^*>0$ such that the problem above has infinitely many weak solutions with negative energy values for any $\lambda\in(0,\lambda^*)$. Paper in conformity with Problems 3 and 4.

18. Kristály A., **Mester Á.**, Mezei I. I., *Sharp Morrey-Sobolev inequalities and eigenvalue problems on Riemannian-Finsler manifolds with nonnegative Ricci curvature*, **COMM. CONTEMP. MATH.**, accepted, 2022. DOI:10.1142/S0219199722500638. [SCIMAGO classification: **D1**] Repository no.: 151684; arXiv: 2107.00512

    Combining the sharp isoperimetric inequality established by Z. Balogh and A. Kristály [*Math. Ann.*, in press, doi:10.1007/s00208-022-02380-1] with an anisotropic symmetrization argument, we establish sharp Morrey–Sobolev inequalities on $n$-dimensional Finsler manifolds having nonnegative $n$-Ricci curvature. A byproduct of this method is a Hardy-Sobolev-type inequality in the same geometric setting. As applications, by using variational arguments, we guarantee the existence/multiplicity of solutions for certain eigenvalue problems and elliptic PDEs involving the Finsler–Laplace operator. Our results are also new in the Riemannian setting. Paper in conformity with Problem 4.

## 3 Conferences, mobilities and talks

- **Kristály A.**:
  - 7th International Conference on Mathematics and Informatics, Targu-Mures, Romania, 2-4 September 2019. Presented: *Brezis-Poincaré-Vázquez inequalities on non-euclidean structures.* [Fully supported by the present project]
  - European Conference on Operational Research, Dublin, Ireland, 23 – 26 June 2019. Presented: *Equality in Riemannian geometric inequalities via optimal mass transportation.* [Fully supported by the present project.]
  - International Conference on Fluids and Variational Methods, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary, 10-14 June 2019. Invited speaker. Presented: *Brunn-Minkowski and entropy inequalities: from Riemannian to sub-Riemannian.* [Talk in the topic of the project.]
  - 23rd IEEE International Conference on Intelligent Engineering Systems, Gödöllő, Hungary, 25-27 April 2019. Presented: *Brezis-Poincaré-Vázquez interpolation inequalities: a geometric approach.* [Presentation of the paper 1)]
  - 2019 Szeged Workshop on Convexity, Bolyai Institute, University of Szeged, Szeged, Hungary, 5-6 April 2019. Presented: *Equality in geometric inequalities via optimal mass transportation: how to get convexity?* [Talk in the topic of the project.]
  - Algebraic Geometry and Differential Topology Seminar, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary, 6 March 2019. Invited speaker. Presented: *Sharp geometric inequalities via optimal mass transportation: Riemannian versus sub-Riemannian.* [Talk in the topic of the project.]
  - Workshop "Geometry and Probability", Fukuoka, Japan, 29 January-2 February 2019. Invited speaker. Presented: *Geometric interpolation inequalities on Heisenberg/Carnot groups.* [Talk in the topic of the project.]
  - Analysis and PDE Seminar, Durham University, 10 December 2021. Presented: *Sharp isoperimetric and Sobolev inequalities in spaces with nonnegative Ricci curvature.* [Talk in the topic of the project.]
  - Corvinus Centre for Operations Research, Corvinus Institute for Advanced Studies, Corvinus University of Budapest, Budapest, Hungary, 8 September 2022. Presented: *Sharp geometric inequalities on non-euclidean settings: an optimal mass transport approach.* [Talk in the topic of the project.]
  - Scientific collaboration: Kristály A. visited the Universitat Bern, Switzerland, 17-30 October 2019. During this period the study of certain Sobolev inequalities on Heisenberg groups with Z. Balogh (Bern) and C. E. Gutierrez (Philadelphia) was initiated, which is materialized by the paper published in *Proc. London Math. Society* (2021).
- **Farkas Cs.**:
  - 7th International Conference on Mathematics and Informatics, Targu-Mures, Romania, 2-4 September 2019. Presented: *Multipolar Hardy inequalities on Riemannian manifolds.* [Talk in the topic of the project.]
  - 30th European Conference on Operational Research, Dublin, Ireland, 23 – 26 June 2019. Presented: *Multipolar Hardy inequalities on Riemannian manifolds.* [Fully supported by the present project.]

- International Conference on Fluids and Variational Methods. June 10-14, 2019, Budapest. [Participant.]
- Atelier de travail en Equations aux Dérivées Partielles, IMAR Bucharest, Romania, 12-14 December 2018. Presented: *A characterization related to Schrödinger equations.* [Talk in the topic of the project.]
- Analysis and PDE Seminar, Durham University, 5 November 2021. Presented: *Compact Sobolev embeddings on non-compact manifolds with applications.* [Talk in the topic of the project.]
- Corvinus Centre for Operations Research, Corvinus Institute for Advanced Studies, Corvinus University of Budapest, Budapest, Hungary, 15 September 2022. Presented: *Compact embeddings on non-compact manifolds with applications.* [Talk in the topic of the project.]

- **Mester Á.**:
  - Winter School on Calculus of Variations and Probability. Toulouse, France, 11-22. February 2019. [Fully supported by the present project; participant.]
  - 2019 IEEE 13th International Symposium on Applied Computational Intelligence and Informatics (SACI) Timisoara, Romania, 29-31. May 2019. Presented paper: A. Mester and A. Kristály. *A bipolar Hardy inequality on Finsler manifolds.*
  - International Conference on Fluids and Variational Methods, Rényi Institute, Budapest, Hungary, 10-14. June 2019. [Participant.]
  - 2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics (SACI) Budapest, Hungary (online conference), 19-21. May 2021. Presented paper: A. Mester and A. Kristály. *Three isometrically equivalent models of the Finsler-Poincare disk.*

## 4 Final notes, concluding remarks

- Scientifically speaking, we were able to provide new connections between functional inequalities, elliptic PDEs, variational and geometric analysis. Our results are cited in highly ranked papers from different mathematicians all around the World (including people from Princeton, Oxford, etc.).
- Another significant byproduct of the project is the establishment of a potentially competitive research group in Geometric and Variational Analysis. Both members, who are my (former) PhD Students, are promising researchers, who are able to work in the aforementioned research fields, by elaborating further innovative methods.

  For instance, Á. Mester just won a Fellowship from the National Research, Development and Innovation Fund of Hungary, and she is going to defend her PhD Thesis very soon (the Thesis has been submitted in June, already obtaining strongly positive reports).
- Summing up, I consider the period of research (2018-2022) has been extremely fruitful, and all the objectives were fully achieved.

Project Leader,
Alexandru Kristály

18 October 2022