# OTKA K124152 final report

## Overview

The main objective of the project was the analysis of trade-off relations in various quantum information theoretic tasks and the mathematical study of the related information measures. In line with this, we have obtained such trade-off results in quantum state discrimination, channel coding, entanglement transformation and multi-state transformations, as well as purely mathematical results on the properties of various quantum divergences (quasi-distances of quantum states).

The publication output of the project at the end of its five-year duration is 74 research papers, of which 53 have already appeared in peer-reviewed journals, 3 in peer-reviewed conference proceedings, and 18 are currently in online available preprint form, being under review at some journal, or just before submission. Many of these papers appeared in leading journals in information theory (IEEE Transactions on Information Theory, 8 papers, with 2 more under review), in mathematical and theoretical physics (Annales Henri Poincare, 1 paper, Quantum, 4 papers, and 2 papers under review at Communications in Mathematical Physics), and in pure mathematics (Advances in Mathematics, 1 paper). All our publications are publicly accessible at the online repository arxiv.org. The complete and up-to-date publication list can be found on our group's website: `http://qi.math.bme.hu/publicationsAll.html`. Below we give a detailed account of a selected list of our results that we consider the most important.

The results of the project were presented at 61 conference and seminar talks, of which 23 were invited conference talks, 27 contributed conference talks, 3 invited tutorials, and 8 invited seminar talks. More details can be found at `http://qi.math.bme.hu/conferencesAll.html`. The largest and most selective conference in (mainly theoretical) quantum information science is the annual Quantum Information Processing (QIP) conference. The PI was invited to give a 3-hour tutorial on entropies at QIP 2019 (`https://jila.colorado.edu/qip2019/program.html`), Péter Vrana presented a talk at QIP 2021 based on [6] and [27] (`https://www.mcqst.de/qip2021/program/thursday.html`), and Mihály Weiner and Zoltán Zimborás presented talks at QIP 2022 based on [25] and [26], respectively (`https://web.cvent.com/event/8adf8248-432b-499c-91e2-63b83ba3f69e/summary`). The most important annual conference series in quantum Shannon theory is Beyond IID in Information Theory, where the group members presented 5 talks (Mihály Weiner in 2021, the PI in 2018, 2019, 2022, and Zoltán Zimborás in 2022, based on the works [25, 24, 23, 5]).

As envisaged in the research proposal, the project has led to fruitful collaborations between the participants, who had not collaborated before the start of the project (despite related interests), with some of them just having had returned after several years abroad. In particular, the PI wrote joint papers with senior participants Péter Vrana [22], Mihály Weiner [25] and Zoltán Zimborás [5], with the latter two papers solving core challenges outlined in the proposal. Following the plans written in the research proposal, the senior participants also put a great emphasis on training young researchers by involving them in the project; this resulted in 2 BSc dissertations (Richárd Simon and Bendegúz Sulyok, supervised by the PI), 4 MSc dissertations (Zoltán Kolarovszki, Gergely Bunth, and Tamás Borsos, supervised by the PI, and Zsombor Szilágyi, supervised by Mihály Weiner), and 4 PhD dissertations are currently in progress

(Gergely Bunth and Gábor Maróti, supervised by the PI, Zsombor Szilágyi, supervised by Mihály Weiner, and Dávid Bugár, supervised by Péter Vrana).

This project and the NKFIH project K129601 with the same PI are strongly related to each other, as explained in the research proposal of the latter, and hence there are also overlaps between the respective reports. On the other hand, the present project ran two years longer, resulting in a correspondingly larger number of publications.

# Results

## I. Hypothesis testing and Rényi divergences

**1. Strong converse exponent and sandwiched Rényi divergences in infinite dimension:** As it was shown in [M. Mosonyi, T. Ogawa, CMP 334(3):1617–1648, 2015], the sandwiched Rényi divergences of two finite-dimensional density operators quantify their asymptotic distinguishability in the strong converse domain. This establishes the sandwiched Rényi divergences as the operationally relevant ones among the infinitely many quantum extensions of the classical Rényi divergences for Rényi parameter $\alpha > 1$. The known proof of this goes by showing that the sandwiched Rényi divergence coincides with the regularized measured Rényi divergence, which in turn is proved by asymptotic pinching, a fundamentally finite-dimensional technique. Thus, while the notion of the sandwiched Rényi divergences was extended recently to density operators on an infinite-dimensional Hilbert space (in fact, even for states of an arbitrary von Neumann algebra), these quantities were so far lacking an operational interpretation similar to the finite-dimensional case, and it has also been open whether they coincide with the regularized measured Rényi divergences. In [20] we fill this gap by answering both questions in the positive for density operators on an infinite-dimensional Hilbert space, using a simple finite-dimensional approximation technique. We also initiate the study of the sandwiched Rényi divergences, and the related problems of the strong converse exponent, for pairs of positive semi-definite operators that are not necessarily trace-class (this corresponds to considering weights in a general von Neumann algebra setting). This is motivated by the need to define conditional Rényi entropies for infinite-dimensional bipartite states, while it is also interesting from the purely mathematical point of view of extending the concept of Rényi (and other) divergences to settings beyond the standard one of positive trace-class operators (positive normal functionals in the von Neumann algebra case). In this spirit, we also discuss the Rényi $(\alpha, z)$-divergences in this setting. In [17] we further extend the above results to states of injective (equivalently, approximately finite-dimensional) von Neumann algebras, and more generally, to states of nuclear $C^*$-algebras. (The paper [20] is under review at Communications in Mathematical Physcis.)

**2. Test-measured Rényi divergences:** One possibility of defining a quantum Rényi $\alpha$-divergence of two quantum states is to optimize the classical Rényi $\alpha$-divergence of their post-measurement probability distributions over all possible measurements (measured Rényi divergence), and maybe regularize these quantities over multiple copies of the two states (regularized measured Rényi $\alpha$-divergence). As mentioned above, a key observation behind the theorem for the strong converse exponent of quantum state discrimination is that the latter quantity coincides with the sandwiched Rényi $\alpha$-divergence when $\alpha > 1$. Moreover, it also follows from the same theorem that to achieve this, it is sufficient to consider 2-outcome measurements (tests) for any number of copies (this is somewhat surprising, as achieving the measured Rényi divergence for $n$ copies might require a number of measurement outcomes that diverges in $n$, in general). In view of this, it seems natural to expect the same when $\alpha < 1$; however, we show in [23] that this is not the case. In fact, we show that even for commuting states (classical case) the regularized quantity attainable using 2-outcome measurements is strictly smaller than the Rényi $\alpha$-divergence (which is unique in the classical case). In the general quantum case this shows that the above "regularized test-measured" Rényi $\alpha$-divergence is not even a quantum extension of the classical Rényi divergence when $\alpha < 1$, in sharp contrast with the $\alpha > 1$ case. On the other hand, we show that for every $\alpha \in (0, 1)$, the regularized test-measured Rényi $\alpha$-divergence exhibits a direct operational interpretation in the context of binary state discrimination; in particular, for $\alpha = 1/2$ it coincides with the Chernoff divergence, the optimal error exponent of symmetric state discrimination.

**3. Composite state discrimination:** The trade-off between the two types of error probabilities in binary state discrimination may be quantified in the asymptotics by various error exponents, depending on the relative weighting of the two types of error probability. In the composite case, where the hypotheses consist of sets of states, any such exponent is upper bounded by the infimum of the corresponding pairwise exponents of discriminating individual members of the two sets. Attainability of this upper bound may depend on the type of exponents considered; whether the problem is classical or quantum; the cardinality and the geometric properties of the sets representing the hypotheses; and also on the dimensionality of the underlying Hilbert space. We clarify this landscape considerably in [25]; in particular, we show that in the quantum case unattainability of the upper bound is the general behaviour for all the exponents, already in finite dimension, even for a simple null-hypothesis and an alternative hypothesis consisting of only two states. This is in sharp contrast with the classical case, where the upper bound is attainable in all previously studied cases. On the other hand, we show that the upper bound may be strict even in the classical case if the system is infinite-dimensional and the alternative hypothesis contains at least countably infinitely many states. We also prove general attainability results, e.g., for classical adversarial and arbitrarily varying hypothesis testing, where we unify and extend several previous results in the literature, and for the quantum case when all states are pure, or the states commute with every state in the opposite hypothesis.

**4. Super-exponential state discrimination:** The above discussed results mainly consider the asymptotics in the i.i.d. setting, where the samples are prepared independently and in the same state. In this setting it is not possible to attain a super-exponential asymptotics for both types of error probability, unless the two states have orthogonal supports, in which case they are perfectly distinguishable using a single copy. In [5] we show that a qualitatively different behaviour can occur when there is correlation between the samples. Namely, we use gauge-invariant and translation-invariant quasi-free states on the algebra of the canonical anti-commutation relations to exhibit pairs of states on an infinite spin chain with the properties that a) all finite-size restrictions of the states have full rank density operators, and b) the type I and the type II error probabilities both decrease to zero at least with the speed $e^{-cn\log n}$ with some positive constant c, i.e., with a super-exponential speed in the sample size $n$. Particular examples of such states include the ground states of the XX model corresponding to different transverse magnetic fields. In fact, we prove our result in the setting of binary composite hypothesis testing, and hence it can be applied to prove super-exponential distinguishability of the hypotheses that the transverse magnetic field is above a certain threshold vs. that it is below a strictly lower value, which is of clear physical interest. (The paper has passed one round of review at Letters in Mathematical Physics, with positive reports.)

**5. Binary channel discrimination and continuity of Rényi divergences:** In the problem of binary quantum channel discrimination, an experimenter has to decide between two candidates for the identity of a quantum channel by feeding quantum states into multiple uses of the same channel, and making measurements on the output. In the case where the experimenter can only use product states as inputs, the problem essentially reduces to a binary i.i.d. state discrimination problem. In this case, the supremum of all type II error exponents for which the optimal type I errors go to zero is equal to the Umegaki channel relative entropy, while the infimum of all type II error exponents for which the optimal type I errors go to one is equal to the infimum of the sandwiched channel Rényi $\alpha$-divergences over all $\alpha > 1$. (The channel divergences are defined as the maximal divergences between the output states of the two channels over the same inputs, possibly including ancilla systems on which the channels act as the identity.) In [21] we prove that these two threshold values coincide, thus proving, in particular, the strong converse property for this problem. Our proof method uses a minimax argument, which crucially relies on a newly established continuity property of the sandwiched Rényi $\alpha$-divergences in their arguments for $\alpha \in (1, 2]$. Motivated by this, we also study the continuity properties of various Rényi divergences and their channel versions, both in the input state/channel pairs and in the Rényi parameter $\alpha$, and we even obtain joint continuity results in all these variables in some cases.

# II. Channel coding and related information measures

**1. Strong converse exponent of classical-quantum channel coding:** The strong converse exponent of a classical-quantum channel $W$ for a given coding rate $R$ (number of messages transmitted per channel use) gives the optimal exponent with which the probability of successful decoding goes to zero when $R$ is larger than the capacity of the channel. In [24] we consider the constant composition version of this problem, which is a refinement of the general coding problem, where each input symbol $x$ of the channel has to be used with a pre-defined frequency $P(x)$. Our main result is that the strong converse exponent can be expressed as $\mathrm{sc}(W, R, P) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha}[R - \chi^*_\alpha(W, P)]$, where $\chi^*_\alpha(W, P) := \inf_\sigma \sum_x P(x) D^*_\alpha(W(x)\|\sigma)$ is the $P$-weighted sandwiched Rényi divergence radius of the channel; here $D^*_\alpha(W(x)\|\sigma)$ is the sandwiched Rényi divergence of the output $W(x)$ of the channel on input $x$ and an arbitrary state $\sigma$ on the output space of the channel. This result further confirms that it is the sandwiched type quantum Rényi divergences that have operational relevance in the strong converse domain of coding problems, and it extends Csiszár's classic result showing that it is the geometric quantity $\chi^*_\alpha(W, P)$ that faithfully quantifies the usefulness of a channel for information transmission with constant composition coding, and not the possibly more intuitive concept of Rényi mutual information. Based on the above result, we also determine the exact strong converse exponent of classical-quantum channel coding with cost constraint.

**2. General study of divergence radii:** As it turns out, the concept of the weighted Rényi divergence radius is naturally connected to an intensively studied concept in matrix analysis, the barycenter (also called Fréchet mean, or Karcher mean). Motivated by [24], we started a systematic study of this concept for various quantum Rényi divergences and more general quantum divergences. In [29] we give a fixed-point equation characterization of the barycenter of finitely many positive matrices with respect to maximal quantum $f$-divergences (also called quantum Hellinger divergences). Continuing this line of study, we show in [30] that general symmetric Kubo–Ando means admit a divergence center (barycenter) interpretation, and we use this result to define natural weighted and multivariate versions of these operator means.

The quantum Jensen–Shannon divergence is the (Umegaki) relative entropy radius of two equally weighted quantum states (more generally, positive semi-definite matrices). It was conjectured more than a decade ago by Lamberti et al. that the square root of this quantity is a genuine metric. We prove this conjecture in [33].

**3. Interconversion of communication resources and correlations:** A central problem in quantum Shannon theory is the optimal rate of interconvertibility of various non-local resources, like shared randomness, entanglement, and more general correlations, or classical or quantum noiseless or noisy communication channels.

To quantify the advantage that a non-signaling resource provides to a noiseless classical channel, one might ask how many extra letters should be added to the alphabet of the channel in order to perform equally well without the specified non-signaling resource. As was observed by Cubitt, Leung, Matthews, and Winter in [PRL 104, 230503, (2010)] and [IEEE Trans. Inf. Th. 57, (2011)], there is no upper bound on the number of extra letters required for substituting the assistance of a general non-signaling resource to a noiseless one-bit classical channel. In contrast, we prove in [13] that if the resource is a bipartite quantum system in a maximally entangled state, then an extra classical bit always suffices as a replacement, showing a very sharp separation between the utility of entanglement and general non-signaling resources in this problem.

In [11], we investigate whether certain non-classical communication channels can be simulated by a classical channel with a given number of states and a given amount of noise. It is proved that any noisy quantum channel can be simulated by a corresponding classical channel with the same amount of noise (in a suitable precise sense). Classical simulations of general probabilistic channels are also studied, and various bounds on signalling dimensions of state spaces are given. (A *Perspective* on this paper has been recently published [Michele Dall'Arno, The signaling dimension of physical systems, Quantum Views 6, 66 (2022)].)

# III. Mathematical study of classical and quantum divergences

**1. Monotonicity of Umegaki's relative entropy:** From the point of view of information theory, the most important mathematical property that a pseudo-distance (divergence) of quantum states has to satisfy is the data processing inequality, or monotonicity, i.e., that the value of the divergence cannot increase if both of its arguments are subjected to the same quantum operation. The latter means a completely positive trace-preserving map for physics applications, while from a mathematical point of view, monotonicity under maps with weaker positivity properties is also interesting, the strongest form of monotonicity being monotonicity under positive trace-preserving (PTP) maps. This strong monotonicity property has been established before for the sandwiched Rényi divergences using interpolation techniques, and from that the same level of monotonicity follows for the Umegaki relative entropy. In [12] we give a completely different proof for this latter monotonicity, by establishing a novel integral representation of the Umegaki relative entropy, which is of independent interest on its own right. We also show that the infimum of any monotone divergence on pairs of quantum states with prescribed trace distance is the same as the corresponding infimum on pairs of binary classical states, and use this to give an improved quantitative concavity bound for the von Neumann entropy.

**2. Barycentric Rényi divergences:** Despite intensive study and applications of quantum extensions of the classical relative entropy (quantum relative entropies), only two such quantities with the desirable properties of monotonicity and additivity on tensor products have been known so far, the Umegaki relative entropy and the Belavkin–Staszewski relative entropy, which are also known to be the smallest and the largest, respectively, among all such quantum relative entropies. (Taking convex combinations of the two yield further such examples, but these are in some sense trivial.) In [22] we introduce a systematic way of obtaining (potentially) new families of quantum relative entropies with the above properties from a given one, using the Kubo–Ando weighted geometric means. In particular, we show that when starting from the Umegaki relative entropy, our construction gives a one-parameter family of new quantum relative entropies, which (at least on state pairs with full rank) interpolates between the Umegaki and the Belavkin–Staszewski relative entropies.

In the classical case, in the limit $\alpha \to 1$ the Rényi $\alpha$-divergences converge to the relative entropy. Interestingly, all the Rényi $\alpha$-divergences can also be recovered from the relative entropy by a simple variational formula; in particular, for $\alpha \in (0,1)$, the Rényi $\alpha$-divergence of two classical states (probability distributions) is the divergence radius of the states corresponding to the case where the divergence is the relative entropy with its arguments interchanged, and the weights are $(\alpha, 1-\alpha)$. In [22] we use an analogous formula to give a systematic way of defining new families of quantum Rényi divergences from two quantum relative entropies, which inherit the monotonicity properties of their parent relative entropies. We show that these barycentric Rényi divergences obtained from the above described one-parameter family of quantum relative entropies are all monotone under PTP maps, and they are different from all previously studied quantum Rényi divergences.

**3. Classical and quantum Wasserstein spaces:** Wasserstein $p$-distances feature in the study of optimal transport as distances on probability distributions on a metric space, and our main line of investigation was the study of the isometries of such spaces. In [14] we describe the group of Wasserstein isometries over the unit interval $[0,1]$ and the real line. As for the real line, we prove isometric rigidity for all positive parameters $p \neq 2$. This is in striking contrast with Kloeckner's result on the quadratic ($p = 2$) Wasserstein space which admits non-trivial and exotic isometries. We find a substantial difference between the real line and the interval as well. Namely, the $p$-Wasserstein space over the interval is rigid for $p > 1$, but for $p = 1$, it has exotic, moreover, mass-splitting isometries. Using this latter phenomenon we give affirmative answers to the questions posed by Kloeckner in 2010 regarding the existence of exotic and mass-splitting isometries in quadratic Wasserstein spaces. In [16] we show the existence of non-trivial isometries in the quadratic case, and isometric rigidity for every $p \neq 2$, for Wasserstein-Hilbert spaces. The key steps here are a metric characterization of Dirac masses and the use of the Wasserstein potential of measures.

In [15] we study the Wasserstein isometries on the space of qubit states. Here, the isometry structure depends on the choice of a cost operator, and we show for a certain natural choice that the isometry group of the quantum Wasserstein space coincides with the orthogonal group $O(3)$, which is both a rigidity and a Wigner-type result. (The paper is under review at J. Math. Anal. Appl.) In [28] we consider a variant of the quantum Wasserstein distance where the optimization is carried out over separable couplings of the states rather than general (potentially entangled) couplings, and discuss the relation of this new quantity to Fisher information and entaglement detection.

# IV. Asymptotic entanglement transformation

A central open problem in the theory of multipartite entanglement is to determine the rate at which a given state can be asymptotically transformed into a given target state by local operations and classical communication. Transformations between bipartite pure states are well understood under various error criteria, and the optimal rates are expressed in terms of information quantities – entanglement measures – namely the von Neumann and Rényi entropies of entanglement. In contrast, the problem of finding the transformation rates when three or more subsystems are involved is wide open. In 2008 Chitambar, Duan and Shi found a connection between tripartite stochastic entanglement transformations and notions in algebraic complexity theory, in particular the unsolved problem of determining the asymptotic complexity of matrix multiplication. Several of our results use or are inspired by this connection and the techniques developed in the study of the exponent of matrix multiplication. The current best upper bound on the exponent is based on a method developed by Coppersmith and Winograd to bound the asymptotic subrank of certain order-three tensors. Any lower bound on the asymptotic subrank corresponds to an entanglement distillation protocol that produces exact Greenberger–Horne–Zeilinger states but only with an arbitrarily small nonzero probability of success. In [39] we propose an entanglement distillation protocol that uses ideas from this method, but results in approximate Greenberger–Horne–Zeilinger states with probability one.

Using a generalization of the Coppersmith–Winograd method to arbitrary orders, in [2] we determine the optimal rate of distilling Greenberger–Horne–Zeilinger states from balanced Dicke states, a class of multi-partite states arising in quantum optics, via stochastic local operations and classical communication.

A remarkable result in bilinear complexity is Strassen's characterization theorem that, in its abstract form, states that a commutative semiring with an Archimedean preorder gives rise to an asymptotic preorder that can be characterized by the asymptotic spectrum, i.e. the set of monotone semiring homomorphisms into the nonnegative reals. In [19] we apply this theorem to entanglement transformations and characterize the trade-off between the rate and the error exponent in the strong converse region. The characterization is in terms of the asymptotic spectrum of LOCC transformations, an axiomatically-defined set of entanglement measures, which in general are not known explicitly. In the bipartite case we completely classify the asymptotic spectrum, identifying its elements as exponentiated Rényi entropies of entanglement with orders between 0 and 1, which leads to an explicit expression for the trade-off curve. In the multi-partite case, in [34] we construct an explicit family of elements of the asymptotic spectrum, which can be viewed as Rényi generalizations of convex combinations of the marginal von Neumann entropies, generalizing recent work by Christandl, Vrana and Zuiddam. The construction involves a multi-partite generalization of the empirical Young diagram measurement, which obeys a large deviation principle. Our formula for the new entanglement measures is given in terms of the rate function, which we determine in [3] in a more general context. (The paper [34] is currently under review at Communications in Mathematical Physics.)

In [4] we find a new way to express the elements of the asymptotic spectrum constructed in [34], in terms of the regularized sandwiched Rényi divergence between tensor powers of the state and certain non-i.i.d. sequences of positive operators. The new expression allows us to simplify the proofs of subadditivity, submultiplicativity, and monotonicity, and when the order parameter is greater than $1/2$, provides an extension to arbitrary convex combinations of entanglement entropies across all bipartitions, that are also subadditive, submultiplicative and monotone. For each of these we construct a corresponding superadditive and supermultiplicative and monotone quantity, and show that they are equal when the reduced state of the state on every subset of the subsystems is proportional to an orthogonal projection. In particular, on the semiring generated by such states the functionals are elements of the asymptotic spectrum. By a known extension result this implies that they admit distinct extensions to the semiring of all pure states, and that these extensions are different from all the known elements of the asymptotic spectrum of LOCC transformations.

In the bipartite case the earliest and arguably most transparent result is about asymptotic transformations with an asymptotically vanishing error. In this limit the entanglement of a state is characterized by a single number, the (von Neumann) entropy of entanglement. In contrast, in the multi-partite realm a similar uniqueness theorem cannot be true due to the non-additivity of the transformation rates. In [36] we find a multi-partite analogue of the uniqueness theorem, identifying axiomatically the relevant information quantities. We show that the rates of transformations between any pair of states with asymptotically vanishing error are characterized by the set of asymptotically continuous additive entanglement measures. In addition, we show that, assuming LOCC-monotonicity on average and additivity, asymptotic continuity is equivalent to an algebraic condition similar to the chain rule for the Shannon entropy.

The appearance of multiple entanglement measures is closely related to the idea of asymptotic irreversibility of entanglement transformations, a fundamental property of multipartite entanglement, and in contrast with bipartite pure states and some other resource theories like thermodynamics. In [9] we introduce this physically motivated notion to the abstract setting of relative bilinear complexity. We define the irreversibility of a tensor that quantifies the loss when transforming unit tensors into many copies of the given tensor and then transforming back to unit tensors (equivalently: reducing a bilinear map to independent scalar multiplications and then reducing independent scalar multiplications to the same map). One can show that the irreversibility of a square matrix multiplication tensor (of any size) is twice the exponent of matrix multiplication. Due to a chain rule property, the irreversibility of a tensor puts a lower bound on any upper bound on the exponent achievable using the given starting tensor, a so-called barrier result. Using this method we improve on previous barrier results on certain approaches to fast matrix multiplication.

# V. Multistate transformations

Relative submajorization is a relation defined on the set of pairs of positive operators, and has applications in hypothesis testing and quantum thermodynamics. We studied an asymptotic relaxation of this relation, comparing tensor powers of each component in the pairs. In the setting of hypothesis testing this corresponds to discriminating many independent and identically distributed copies (in particular in the strong converse regime), while in quantum thermodynamics it describes many non-interacting copies of the system, coupled to a thermal bath at a common temperature. While the componentwise direct sum and tensor product together with relative submajorization gives rise to a preordered semiring, Strassen's theorem is not directly applicable because the semiring does not satisfy a crucial Archimedean property. This motivated our work [37], where we proved a generalization of Strassen's characterization theorem to certain non-Archimedean preordered semirings. Under the weaker polynomial growth condition, one can still define the asymptotic preorder and we found conditions for the characterization property to hold. Based on this result, in [27] we characterized asymptotic transformations between pairs of operators in terms of the asymptotic spectrum, and classified its elements, identifying them as the exponentiated sandwiched Rényi divergences of orders greater than 1. This led to a new proof of the strong converse exponent in quantum hypothesis testing, as well as a trade-off between the error exponent and the rate for work-assisted Gibbs-preserving operations in quantum thermodynamics. Subsequently, in [6] and [7] we managed to partially generalize these results to composite hypothesis testing as well as group symmetric hypothesis testing problems and state transformations by time-translation covariant Gibbs-preserving maps (also known as thermal processes). In connection with the latter problem we identified new so-called second laws of thermodynamics that apply to systems of any size, which put constraints on the evolution of states with coherence as well as athermality, even in the presence of catalysts. This was made possible by constructing explicit elements of the asymptotic spectrum in terms of the sandwiched Rényi divergence and Kubo–Ando (and multivariate) operator geometric means. In addition, our partial understanding of the relevant asymptotic spectrum led to a new two-parameter family of monotone quantum Rényi divergences.

# VI. Zero-error communication

Strassen's theory of asymptotic spectra can be applied to other problems in information theory as well. The zero-error capacity of a channel is a parameter of its confusability graph, which led to the notion of the Shannon capacity of a graph. Zuiddam showed that the set of isomorphism classes of finite simple undirected graphs with disjoint union, strong product and the cohomomorphism preorder forms a preordered semiring satisfying the conditions in Strassen's theorem, and that the abstract asymptotic subrank is equal to the zero-error capacity. It follows that the capacity is equal to the minimum of the values of all parameters in the asymptotic spectrum of graphs, i.e. the set of real-valued graph parameters that are in this sense semiring-homomorphic and monotone, which include well-known upper bounds on the Shannon capacity such as the Lovász number and the fractional Haemers bounds. In [35] we make significant progress in understanding the asymptotic spectrum of graphs, by showing that it has a convex structure, allowing us to construct uncountably many new elements. Among these we find graph parameters that for certain graphs provide better upper bounds on the Shannon capacity than any of the previously known ones.

The asymptotic spectrum of graphs does not only provide a characterization of the Shannon capacity, but also the existence of asymptotic cohomomorphisms, which can be seen as block-encodings of strings over alphabets equipped with a distinguishability relation. Valid encodings are required to map distinguishable words over one alphabet to distinguishable words, which includes as special cases the zero-error capacity problem as well as a coding problem for an i.i.d. source with an ambiguous alphabet proposed and solved by Körner. In [32] we extend this problem to dependent information sources and propose a notion of the asymptotic equipartition property that is suitable for an arbitrary source with a partially distinguishable alphabet. We show that the new property is satisfied by any Markov source, which in particular implies an operational interpretation of the graph entropy rate of a Markov chain with respect to any confusability graph. (The paper is under review at IEEE Transactions on Information Theory.)

The zero-error communication problem has several generalizations in quantum information theory, depending on whether the transmitted information is classical or quantum, and whether an additional source of entanglement can be used in the process. It is known that the entanglement-assisted classical or quantum zero-error capacities can be characterized in terms of the asymptotic spectra of semirings of (noncommutative) graphs with the entanglement-assisted cohomomorphism preorder, and that any graph parameter that is multiplicative, additive, and monotone under entanglement-assisted cohomomorphisms can be extended to noncommutative graphs with the same properties, by a result of Li and Zuiddam. However, the corresponding questions for the unassisted capacities of noncommutative graphs (relevant for the problem of zero-error classical or quantum communication over a noisy quantum channel) are not answered by Strassen's theory or its known extensions. In [38] we study the extension problem in this setting, and find that if an element of the asymptotic spectrum of graphs has a multiplicative, additive, monotone extension to the semiring of noncommutative graphs, than it has infinitely many of them. More precisely, any extension has an exponent determining its values on the confusability graphs of noiseless quantum channels, and the set of possible exponents is either empty or an unbounded interval. We determine the set of admissible exponents for the fractional clique cover number, the projective rank, the Lovász number, and the fractional Haemers' bound over the complex numbers. (The paper is under review at IEEE Transactions on Information Theory.)

# VII. Quantum information theory and statistical physics

**1. Tensor network representations:** In [8] we study tensor network representations of many-body quantum states from the point of entanglement transformations and tensor restrictions. Projected entangled pair states and their generalizations form important ansatz classes that provide good approximations to low-energy states of local many-body Hamiltonians. Such a representation, known as a tensor network state, can be viewed as the result of a local transformation applied to a pattern formed by maximally entangled bipartite states. We show that degeneration, a basic tool in the construction of matrix multiplication algorithms, gives rise to a modified ansatz class that provides a similarly efficient representation of states in the closure of the set of tensor network states with a given bond dimension (with a linear overhead in the system size, as opposed to the exponential cost of increasing the bond dimension).

**2. Extendibility of Werner states:** State extension problems have played a prominent role in entanglement theory since its very beginning. The so-called *symmetric extendibility* turned out to be a particularly useful concept, that captures how much a bipartite state can be shared between parties: A bipartite state $\rho_{AB}$ shared between Alice and Bob is said to be $(n_\mathrm{A}, n_\mathrm{B})$-extendible if there exists a state shared between $n_\mathrm{A}$ number of Alices and $n_\mathrm{B}$ number of Bobs, such that reducing the state to any Alice-Bob pair yields back the original state $\rho_{AB}$. Despite the prominent role extendibility plays in entanglement theory, it has been calculated analytically only for a few families of entangled states. Notably even for Werner states only the analytic condition for one-sided, i.e., $(1, n)$-extendibility was previously derived, and the general two-sided problem has been solved just for a couple of specific extension sizes. In Ref. [18], we determine necessary and sufficient conditions for the two-sided $(n_\mathrm{A}, n_\mathrm{B})$-extendibility of Werner states for arbitrary values of $n_\mathrm{A}$, $n_\mathrm{B}$ and local dimension $d$. The interplay of the unitary symmetry of these states and the inherent bipartite permutation symmetry of the extendibility scenario allows us to map this problem into the ground state problem of a highly symmetric spin-model Hamiltonian. The eigenvalues of this Hamiltonian are labeled by triples of Young diagrams that must be compatible with each other w.r.t. the Littlewood–Richardson product of diagrams. By utilizing the dominance order of Young diagrams in the Littlewood–Richardson product, we can reduce the number of variables and solve the ground state problem exactly.

**3. Quantum information theory in fermionic systems:** Fermionic systems, such as fermionic ultra-cold atoms or electrons, are often used in current quantum technologies. When examining possible quantum information theory protocols with fermions, it should be noted that the mathematical description of multiparticle fermionic systems differs from that of conventional quantum spin systems. On the one hand, in the case of fermions, the embedding of subsystems into larger systems does not follow the traditional tensor product structure, and on the other hand, the so-called parity superselection rule must also be applied for fermions. Due to the experimental relevance of such systems, the development of quantum information theory in fermionic systems is an area of active research. In [10] we generalize the notion of the relative entropy of entanglement to fermions, taking into account parity superselection, and derive a closed formula for this quantity for subsystems containing two modes. We also consider applications of these findings in quantum chemistry. In [31] we examine the asymptotics of the entanglement negativity generalized to fermions for the ground states of several quasi-periodic and randomly modulated fermion chains. In [1] we use fast oscillating non-analytic matrix functions as Toeplitz symbols to construct families of non-gauge-invariant quasi-free states for which the local Rényi and von Neumann entropies diverge sublogarithmically with the subsystem size. These are the first translation-invariant states presented in the literature that have such slowly diverging local entropies.

Fermionic systems also offer promising candidates for experimentally implementable and verifiable quantum supremacy schemes, i.e., tasks where a quantum computer significantly outperforms any classical computer. The famous Google quantum supremacy experiment of 2020 and the boson sampling experiment conducted last year at the University of Science and Technology of China fall into this class. Compared to the random quantum gate sampling used in the Google experiment, the random distribution generated by boson sampling has much more structure, and using this, the unitary operation generating the input state in such an experiment can be verified by a polynomial number of measurements in the number of bosonic modes. However, boson sampling is tailored for platforms utilizing photonic modes, i.e., it cannot be implemented without a significant overhead on a quantum computer with a qubit architecture. In [26] we introduce a fermion sampling protocol that is similar to boson sampling in the sense that it can be efficiently verified by a number of measurements scaling polynomially with the number of modes, and, moreover, it has the extra feature that, using the Jordan–Wigner transformation, it can also be efficiently implemented on a qubit based quantum computer.

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