Final report on the OTKA Project **K-124138**

Crossed modules of Hopf monoids

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This is the final report on an OTKA project that was planned for 48 months, but was — somewhat unexpectedly — terminated after 29 months because the PI's leaving academia on May 1 2020.

Below we demonstrate which ones of the planned aims got reached, and what additional results were achieved. Although it exceeds the time proportional ratio of the original proposal, it will remain a torso in a sense. In the subsequent paragraphs we also comment on the potential directions of further research that our work opened, and on possible applications of our findings. We hope that some of our posterity will continue the work that we have no chance to complete.

We use acronyms when referring to papers in the literature, but numbers for the particular papers born from the current project so to distinguish them.

Crossed modules of monoids.

The central topic of the project was the extension of the classical theory of crossed modules from groups to Hopf monoids. Although in the original proposal we did not plan to go above that level of generality, our research revealed that the most efficient setting is to work in the much broader context of *monoids in symmetric monoidal categories*. This does not only cover ordinary monoids (i.e. sets with an associative and unital multiplication) and bimonoids (that are monoids in categories of comonoids), but also small categories and much more. This approach allows to use the highly powerful tool of distributive laws [1].

As we had aimed at, we obtained equivalences between appropriately defined crossed modules, suitable internal categories, and certain simplicial objects, over monoids in symmetric monoidal categories (relative to a chosen class of spans). In the group case, such equivalences make crossed modules a valuable tool in a wide range of applications.

Below we present a bit more details about the main steps, and the most innovative aspects of our work.

— Conventionally, internal categories are defined in terms of pullbacks. The categories of general comonoids, however, — in which bimonoids can be seen as monoids — may not have pullbacks. In order to cover also such categories, in [1] we introduced pullbacks *relatively to some admissible class of spans*. In particular, the class of all spans is admissible. Assuming that the relative pullbacks exist for those cospans whose 'legs' are in the admissible class, we defined a monoidal category of those spans whose 'legs' are in the admissible class. The monoids in this category can be seen as internal categories *relatively* to this class. The classical 'non-relative' case corresponds to the class of all spans.

— It was known from [Pa] that naively defined crossed modules over arbitrary ordinary monoids are not equivalent to internal categories of monoids. Such an equivalence holds exactly if an additional condition, the so-called *Schreier condition* is added to the definition of crossed module (which becomes redundant if the monoids are groups).

In [2] we extended the Schreier condition to monoids in symmetric monoidal categories; and used it do define their crossed modules (relatively to a given admissible class of spans). Applying our definition to bimonoids, the generalized Schreier condition becomes redundant for Hopf monoids. Applying our definition to small categories, it becomes redundant for groupoids. In this way our definition covers crossed modules of Hopf monoids in [FV], and crossed modules over groupoids in [BI].

In [2], for any symmetric monoidal category C, and any admissible class of spans in C, we proved an equivalence between

- ∗ the category of relative crossed modules over monoids in C;
- ∗ the category of relative categories in the category of monoids in C.

In the symmetric monoidal category of comonoids we can choose a canonical admissible class of spans (in a bit sloppy way, the adjective 'relative' is omitted if it were meant with respect to this canonical class). Thereby the above equivalence reduces to an equivalence between

- ∗ the category of crossed modules over bimonoids;
- ∗ the category of categories in the category of bimonoids.

This equivalence restricts to an equivalence between

- ∗ the category of crossed modules over Hopf monoids;
- ∗ the category of Cat^1-Hopf monoids

as in [FV].

— The simplicial nerve of an internal category in the category of groups is a simplicial group distinguished by the property that its Moore complex is concentrated in degrees in 0 and 1. Moreover, simplicial groups with this property provide a further equivalent description of crossed modules. For simplicial monoids in general symmetric monoidal categories, however, the Moore complex may not exist.

In [3] we introduced the *Moore length* of a simplicial monoid also in those cases where the Moore complex does not exist. We proved that whenever the Moore complex exists, its length is equal to our more general Moore length (justifying the terminology). We also proved that, for any symmetric monoidal category C, and any admissible class of spans in C, the above equivalent categories of relative crossed modules, and of relative categories of monoids, are equivalent furthermore to

- ∗ the category of those simplicial monoids in C whose Moore length is 1 and which behave well with respect to the admissible class of spans in a suitable sense.

This leads to an equivalence between crossed modules of bimonoids, and simplicial bimonoids of Moore length 1 (which restricts to an analogous equivalence for cocommutative Hopf monoids).

Higher crossed modules.

Originally we planned to define higher crossed modules (of Hopf monoids), and obtain analogous equivalences for them, by iterating the above considerations similarly to the group case. However, we had to learn that such a simple process is not possible for monoids in general symmetric monoidal categories. This is because our definition of crossed module only works for monoids, and we are not able to interpret crossed modules themselves as monoids. Some alternative way had to be found.

We worked on the extension of the theory in [1–3] as follows. We constructed a family of 2-categories $E(n)$ for all non-negative integers n together with two pseudo-naturally equivalent 2-functors $E(n+1) \to E(n)$. Their iterates $E(n) \to \cdots \to E(0)$ send smartly chosen objects to crossed n-cubes, and n-fold categories, respectively, in the various examples. Thus the equivalence of very general crossed n-cubes and n-fold categories follows immediately from pseudo-natural equivalence of our 2-functors.

This theory has a much wider applicability than categories of monoids. Among others, it is suitable to describe (higher) crossed modules of Lie- and Jordan algebras.

These results have not been published because the application to examples of interest has not been finished.

In addition to that was planned in the original proposal, we obtained the following further results.

The formal theory of multimonoidal monads.

A recent paper [Ag] studied so-called *multimonoidal monads* on multimonoidal categories. This means categories with $p+q$ compatible monoidal structures, and monads which are monoidal with respect to p ones of the monoidal structures and opmonoidal with respect to the remaining q ones in a compatible way. It is proven in [Ag] that — under some minor assumptions — the multimonoidal structure of the base category lifts to the Eilenberg-Moore category of the monad in such a way that the forgetful functor is compatible with q ones of the monoidal structures and its left adjoint is compatible with the remaining p ones. However, no deeper explanation of this fact is given.

We felt strongly urged to derive their result from higher principles, by developing a formal theory of multimonoidal monads in [4] as follows. We defined multimonoidal objects and multimonoidal monads in any symmetric strict monoidal 2-category as the 0-cells of suitable double categories of 'squares' in [Eh] and of 'monads' in the double category of squares in [F1], respectively. We found criteria for Ehresmann's double category of squares in a 2-category to admit Eilenberg-Moore construction in the sense of [F2]. The arising right 2-adjoint of the inclusion double functor sends multimonoidal monads to their multimonoidal Eilenberg-Moore object in such a way that the forgetful 1-cell is monoidal for q ones of the monoidal structures and its left adjoint is monoidal for the remaining p ones.

Applying this to the symmetric strict monoidal 2-category of categories, functors and natural transformations, we re-obtain the main result of [Ag].

The Gray monoidal product of double categories.

The above formal theory was restricted to multimonoidal monads in symmetric strict monoidal 2-categories, although many important examples (e.g. the bicategory of bimodules playing crucial role in Hopf algebra theory) goes beyond this class: they are only equivalent to some Gray monoid. So it would be desirable to extend the formal theory of multimonoidal monads to symmetric Gray monoids.

As we explained above, the formal theory of multimonoidal monads cannot be formulated in the realm of bicategories, it necessarily involves (pseudo-) double categories. While the double categories of 'squares' [Eh] and of 'monads' [F1] in a symmetric strict monoidal 2-category are symmetric strict monoidal double categories, those for a (symmetric) Gray monoid should be (symmetric) *Gray double monoids* if such a notion were available in the literature. In [5] we filled this conceptual gap.

A strict monoidal 2-category is, in other words, a monoid in the category $\mathsf{2cat}$ of 2-categories and 2-functors with respect to the Cartesian product $\times$. Gray monoids, on the other hand, are monoids in $\mathsf{2cat}$ with respect to another monoidal product $\otimes$ due to [G]. Analogously, a strict monoidal double category is a monoid in the category dbl of double categories and double functors with respect to the Cartesian product $\times$. Thus our job in [5] was to find another monoidal product $\otimes$ on dbl such that the resulting monoids possess the properties that Gray double monoids are expected to have.

In the same spirit as in [G], we constructed the new monoidal structure on dbl via the corresponding *closed* structure. In this way it consisted of the following main steps.

— The most creative part was finding the suitable candidates of internal hom double categories $[\mathsf{C},\mathsf{D}]$ (for any double categories C,D). In our construction the 0-cells are the double functors $\mathsf{C}\to\mathsf{D}$. For playing the roles of the horizontal and vertical 1-cells, we used so-called horizontal and vertical pseudo-transformations; together with their modifications to act as the 2-cells. We extended it to a functor $[-,-]$.

— We proved the representability of the functor $[\mathsf{C},[\mathsf{D},-]]$ for any double categories C,D. The representing object defines the new monoidal product $\mathsf{C}\otimes\mathsf{D}$ and, by universality, $\otimes$ as a bifunctor.

— We checked the coherence of the monoidal product $\otimes$ by constructing MacLane's associativity and unitality natural isomorphisms. This proves that $(\mathsf{dbl}, \otimes)$ is a (closed) monoidal category.

— The ambiguity in the choice of the internal hom double categories results in some ambiguity in the resulting monoidal structure. In order to justify our choice, we systematically compared our $\otimes$ with several well-established structures. That is, we proved the monoidality of the following functors.

* The identity functor $(\mathsf{dbl}, \times) \to (\mathsf{dbl}, \otimes)$.
* The functors $(\mathsf{dbl}, \otimes) \to (\mathsf{2cat}, \otimes)$ sending double categories to their horizontal – or vertical – 2-categories.
* The square (or quintet) construction functor $(\mathsf{2cat}, \otimes) \to (\mathsf{dbl}, \otimes)$ in [Eh].
* The functor $(\mathsf{dbl}, \otimes) \to (\mathsf{dbl}, \otimes)$, sending a double category to the double category of its monads in [F1].

Since the publication of [5], Bojana Femić also checked (but has not published) the monoidality of the functor $(\mathsf{2cat}, \otimes) \to (\mathsf{dbl}, \otimes)$ which sends a 2-category to the double category obtained by adding identity horizontal — or vertical — 1-cells. In [Mo] a model structure was constructed on dbl which was shown to be compatible with $\otimes$ in a suitable sense.

In [5] we also gave a quite explicit description of the monoids in $(\mathsf{dbl}, \otimes)$, that should be interpreted as *Gray double monoids.* This would open the door for the extension of the formal theory of multimonoidal monads to symmetric Gray monoids. This is left to someone else in the future.

BiHom bimonoids viewed as bimonoids.

The work in [6] can be seen as part of a much bigger project. In recent years, lead by diverse motivations, several different generalizations of Hopf algebra have been proposed. Their similar features naturally raise the question whether they are instances of the same, more general notion. In [BL] several examples, such as groupoids, Hopf monoids in braided monoidal categories, Hopf algebroids over central base algebras, weak Hopf algebras and Hopf monads on autonomous monoidal categories [BV] were unified as Hopf monoids in certain *duoidal categories* (introduced in [AM] under the name *2-monoidal* category). In [Bö] also Hopf group algebras [Tu], Hopf categories [Ba] and Hopf polyads [Br] were shown to fit this framework. The aim of [6] was an interpretation of the (unital and counital) BiHom-bimonoids in [Gr] as bimonoids in a category with some generalized duoidal structure.

The definition of a BiHom-bimonoid includes an underlying object together with four endomorphisms. Generalizing a construction in [CG], in [6] we constructed a duoidal category whose bimonoids are precisely those BiHom-bimonoids for which all of these endomorphisms are invertible.

In order to treat the general case, however, we had to go beyond the known framework of duoidal category. We had to give up the invertibility of the coherence natural transformations and use the *unbiased* variants of monoidal category. Although there is a meaningful theory of duoidal categories with one lax and one oplax monoidal structure, for the description of BiHom-structures neither the lax nor the oplax variant looks suitable but rather a certain mixture of both. For the definition of BiHom-monoids, associativity constraints of the oplax type are needed. For defining their units, however, we need unit constraints of the lax type.

Axiomatizing this mixed situation, in [6] we introduced so-called $\mathsf{Lax}_0\mathsf{Oplax}_+$-monoidal categories, which have n-fold monoidal products for any non-negative integer n such that the monoidal products of positive number of factors are oplax coherent while the 0-fold monoidal product is lax coherent. We defined monoids in $\mathsf{Lax}_0\mathsf{Oplax}_+$-monoidal categories. Dually, we introduced so called $\mathsf{Lax}^+\mathsf{Oplax}^0$-monoidal categories, again with n-fold monoidal products for any non-negative integer n such that the monoidal products of positive number of factors are lax coherent while the 0-fold monoidal product is oplax coherent. We defined comonoids in $\mathsf{Lax}^+\mathsf{Oplax}^0$-monoidal categories. Finally we defined $\mathsf{Lax}_0^+\mathsf{Oplax}_+^0$-duoidal categories with compatible $\mathsf{Lax}^+\mathsf{Oplax}^0$- and $\mathsf{Lax}_0\mathsf{Oplax}_+$-monoidal structures. Monoids in them were shown to constitute a $\mathsf{Lax}_0\mathsf{Oplax}_+$-monoidal category and, dually, comonoids in them were shown to constitute a $\mathsf{Lax}^{+}\mathsf{Oplax}^{0}$-monoidal category. So we could define bimonoids in them as comonoids in the category of monoids; equivalently, as monoids in the category of comonoids.

Generalizing the above duoidal category in the case of invertible endomorphisms, we constructed a $\mathsf{Lax}_{0}^{+}\mathsf{Oplax}_{+}^{0}$-duoidal category whose monoids, comonoids and bimonoids, respectively, are the unital BiHom-monoids, counital BiHom-comonoids and the unital and counital BiHom-bimonoids in the sense of [Gr].

Integral theory of Hopf monoids.

The work in [7] is also related to the above mentioned unifying description of Hopf algebra-like structures in [BL] as Hopf monoids in the duoidal endohom category of a naturally Frobenius map pseudo-monoid in a suitable monoidal bicategory.

The theory of *integrals* is a powerful tool in the study of Hopf algebras. Their existence and properties contain information about the algebraic structure. It was our long term plan to extend the theory of integrals to Hopf monoids in the duoidal endohom category of a naturally Frobenius map monoidale. This setting is suitable for this kind of study because by [BL] the antipode is available (which is not the case for more general base pseudo-monoid). The so obtained results not only would unify all existing theorems on the various examples; but also prove, at the same time, many new results in the cases whose integral theory has not yet been separately studied. The reported research in [7] can be seen as the first step of this programme, that should be completed by someone else in the future.

Maschke's classical theorem [Ma] sates that the group algebra kG of a finite group G over an arbitrary field k is semisimple if and only if the characteristic of k does not divide the order of G. A generalization to Hopf algebras in [LS] says that a Hopf algebra H over a field is semisimple if and only if it possesses a normalized integral; that is, an H-module section of the counit. A Hopf algebra over a field turns out to be semisimple if and only if it is separable; that is, its multiplication admits a bimodule section. This is no longer true for Hopf algebras over more general commutative base rings. In this more general situation it is the separability of a Hopf algebra that becomes equivalent to the existence of a normalized integral. Maschke type theorems — relating separability to the existence of normalized integrals — were proved individually for many generalizations of Hopf algebra. The aim of [7] was the unification of all these results into a single theorem.

We defined in [7] integrals and cointegrals for bimonoids in arbitrary duoidal categories. For Hopf monoids in the duoidal endohom category of a naturally Frobenius map pseudo-monoid in a monoidal bicategory, the existence of a normalized integral was shown to be equivalent to the separability of the constituent monoid. Dually, the existence of a normalized cointegral was sown to be equivalent to the coseparability of the constituent comonoid.

Let us stress that the statements of both main Maschke-type Theorems in [7], about separability and coseparability, are apparently dual of each other. However, no duality principle is known that would allow us to derive one of them from the other. They were proved independently, by basically different steps.

Our findings in [7] cover the existing Maschke-type theorems on Hopf monoids in braided monoidal categories, weak Hopf algebras, Hopf algebroids over central base algebras and Hopf monads on autonomous monoidal categories, while they yield new results for Hopf categories.

The next question to address here would be about the relation between the existence of non-singular integrals and the Frobenius property of the constituent (co-) monoid. The fact that such (Larson-Sweedler-type) theorems exist for several ones of the covered examples — see e.g. the brand new paper [Bu] about Hopf categories of [Ba] — suggests that this project should be feasible (although not by us).

REFERENCES

[AM] M. Aguiar and S. Mahajan, *Monoidal functors, species and Hopf algebras*, volume 29 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2010.

[Ag] M. Aguiar, M. Haim and I. López Franco, *Monads on higher monoidal categories*, Appl. Categ. Structures 26 no. 3 (2018) 413-458.

[Ba] E. Batista, S. Caenepeel and J. Vercruysse, *Hopf Categories*, Algebr. Represent. Theory 19 no. 5 (2016) 1173-1216.

[Bö] G. Böhm, *Hopf polyads, Hopf categories and Hopf group monoids viewed as Hopf monads*, Theor. and Appl. of Categories 32 no. 37 (2017) 1229–1257.

[BL] G. Böhm and S. Lack, *Hopf comonads on naturally Frobenius map-monoidales*, J. Pure Appl. Algebra 220 no. 6 (2016) 2177–2213.

[BI] R. Brown and İ. İçen, *Homotopies and Automorphisms of Crossed Modules of Groupoids*, Appl. Categ. Structures 11 no. 2 (2003) 185-206.

[Br] A. Bruguières, *Hopf Polyads*, Alg. Represent. Theory 20 no. 5 (2017) 1151–1188.

[BV] A. Bruguières and A. Virelizier, *Hopf monads*, Adv. in Math. 215 no. 2 (2007), 679-733.

[Bu] M. Buckley, T. Fieremans, C. Vasilakopoulou and J. Vercruysse, *A Larson-Sweedler Theorem for Hopf V-Categories*, arXiv:1908.02049v2.

[CG] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra 39 (2011) 2216–2240.

[Eh] C. Ehresmann, *Catégories structurées III. Quintettes et applications covariantes*, in: "Séminaire Ehresmann. Topologie et géométrie différentielle" 5 (1963) 1-21.

[F1] T.M. Fiore, N. Gambino and J. Kock, *Monads in double categories*, J. Pure Appl. Algebra 215 no. 6 (2011) 1174–1197.

[F2] T.M. Fiore, N. Gambino and J. Kock, *Double adjunctions and free monads*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 53 no. 4 (1012) 242-306.

[FV] J.M. Fernández Vilaboa, M.P. López López and E. Villanueva Novoa, Cat^1-*Hopf Algebras and Crossed Modules*, Comm. Algebra 35 no. 1 (2006) 181-191.

[Gr] G. Grazianu, A. Makhlouf, C. Menini and F. Panaite, BiHom-*Associative Algebras,* BiHom-*Lie Algebras and* BiHom-*Bialgebras*, SIGMA 11 (2015), 086, 34 pages.

[G] J.W. Gray, *Formal category theory: adjointness for 2-categories*, Lecture Notes in Mathematics 391 Springer-Verlag, Berlin-New York, 1974.

[LS] R.G. Larson and M.E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. 91 (1969), 75-93.

[Ma] H. Maschke, *Über den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen*, Math. Annalen 50 (1898), 492-498.

[Mo] L. Moser, M. Sarazola and P. Verdugo, *A* 2Cat-*inspired model structure for double categories*, preprint Apr 2020, available at arXiv:2004.14233.

[Pa] A. Patchkoria, *Crossed semimodules and Schreier internal categories in the category of monoids*, Georgian Math. J. 5(6), 575-581 (1998).

[Tu] V. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*, arXiv:math/0005291.

REFERENCES TO PAPERS BORN FROM THE PROJECT

[1] G. Böhm, *Crossed modules of monoids I. Relative categories*, Appl. Categ. Structures 27 no. 6 (2019) 641-662.

[2] G. Böhm, *Crossed modules of monoids II. Relative crossed modules*, Appl. Categ. Structures 28 no. 4 (2020) 601-653.

[3] G. Böhm, *Crossed modules of monoids III. Simplicial monoids of Moore length 1*, Appl. Categ. Structures Online First. DOI

[4] G. Böhm, *The formal theory of multimonoidal monads*, Theor. and Appl. of Categories 34 no. 12 (2019), 295-348.

[5] G. Böhm, *The Gray monoidal product of double categories*, Appl. Categ. Structures 28 no. 3 (2020) 477-515.

[6] G. Böhm and J. Vercruysse, *BiHom Hopf algebras viewed as Hopf monoids*, AMS Contemp. Math. in press. arXiv:2003.08819

[7] G. Böhm, *Masckke type theorems for Hopf monoids*, Theor. and Appl. of Categories in press. arXiv:2004.11519