Deatailed report about the research conducted by the participants of the project K123782

The research conducted by the members of this project can be naturally divided into tree topics: Billiards and related models, Fractals and Related Models and Fractal networks. The description of our results are detailed according to this categorization.

1 Billiards and related models

Mathematical billiards study the motion of a point particle in a domain Q with piecewise smooth boundary ∂Q. The particle travels uniformly (with unit speed) inside Q, and bounces off elastically (the angle of reflection equals the angle of incidence) at the boundary ∂Q. The dynamics can be studied either in continuous time or in discrete time (from collision to collision), resulting in the billiard flow and the billiard map, respectively. Depending on the shape of Q, billiards may exhibit a wide range of dynamical phenomena. In particular, *dispersing billiards* are obtained by removing smooth and strictly convex scatterers C_i from the d dimensional Euclidean plane $\mathbb{R}^d$ or the flat torus $\mathbb{T}^d$ (where $d \geq 2$). In this case both the map and the flow have strong chaotic and ergodic properties with respect to their natural absolutely continuous invariant measures. We refer to [28] for further details.

The main research goal of our project was to apply dispersing billiards (mostly of dimension $d = 2$) to study various phenomena arising from physics and geometry. For the investigation of some of the problems we considered other dynamical systems, in particular expanding maps of the interval, which resemble dispersing billiards but are somewhat less complicated.

1.1 Lorentz gas

The Lorentz gas is a popular model of diffusion obtained from a dispersing billiard with periodic scatterer configuration on $\mathbb{R}^2$ (which by periodicity can be reduced to a dispersing billiard on $\mathbb{T}^2$). Of particular interest is the case when a single circular scatterer of radius $\rho < \frac{1}{2}$ is placed about each point of the integer lattice $\mathbb{Z}^2$, which necessarily produces *infinite horizon*: the free flight length between consecutive collisions can be arbitrary large. This results in superdiffusive behaviour: a limit theorem for the position of the particle with a non-standard $\sqrt{t \log t}$ scaling, proved by Szász and Varjú in [61], which was the starting point of many further works on the long time asymptotic for fixed scatterer size ρ. In a different direction, Marklof and Strömbergson in [45] studied the Boltzmann-Grad limit of the Lorentz gas which corresponds to a limit process that arises from the billiard dynamics when the scatterer size ρ tends to 0. Marklof and Tóth in [46] then studied the large time asymptotics of the process obtained in [45] and proved a limit theorem, again with $\sqrt{t \log t}$ scaling. Our work [3] in a sense interpolates between these two class of results: we prove a limit law for certain situations when time $t \to \infty$ and scatterer size $\rho \to 0$ can be scaled simultaneously.

In the paper [4] we study decay of correlations for flows and prove in particular the sharp upper bound $\frac{1}{t}$ for the time correlations of dispersing billiard models on $\mathbb{T}^2$ corresponding to infinite horizon Lorentz gases. The somewhat complementary work [9] investigates statistical properties for slowly mixing flows, specifically we prove the (functional) central limit theorem, moment estimates and iterated versions of these results for the famous Lorenz attractor (where Lorenz is not to be confused with Lorentz).

1.2 The ball-piston model and related questions

An important motivation for studying billiards is to find appropriate models for various statistical physics phenomena, see the surveys [60] and [7] for further discussion. We mentioned (super)diffusion in the previous subsection, another important problem is to derive the laws of heat conduction. To this end, a two step strategy was proposed in [34]. The first step is to obtain, from an appropriate time scale of a mechanical model, a mesoscopic Markovian dynamics of the slow variables, the energies of the particles; while the second step is to derive Fourier's law of heat conduction from the mesosocpic model. In [6] we introduced a billiard ball-piston system as a mechanical model for which this strategy could be implemented. The rigorous completion of the first step of the strategy for the ball-piston model, which we think is within close reach, has been an important motivation for our research. Key technical ingredients, achieved in [10], were to develop a flow version of the standard pair technique, introduced initially in [27] in the context of the billiard map, and to express the breakthrough result of [2] on exponential decay of correlations for dispersing billiard flows using this language. A related problem was to clarify various notions of Hölder continuity that are relevant in this dynamical setting, which was accomplished in [62].

1.3 Coupled chaotic maps

Coupled systems consist of a graph of interactions to each vertex of which a dynamical system is associated. These individual systems are called sites and the strength of their interaction – which occurs along the edges of the graph – is scaled by some parameter $\varepsilon > 0$. Our research focused on the case when the individual systems are strongly chaotic (such as expanding maps of the interval) and the interaction is mean field, that is, we consider a complete graph with N nodes. For small values of ε, the coupled system has a unique ergodic absolutely continuous invariant measure, for larger values of the coupling strength, however, other types of behavior arise. In particular, for the specific case of coupled doubling maps, in [33] Fernandez discussed the ergodicity breaking phenomena (emergence of multiple absolutely continuous invariant measures), rigorously for $N = 3$ sites and numerically for higher values of N. In [53], on the one hand, we reconsidered the $N = 3$ case giving an alternative, symmetry-based approach to ergodicity breaking, and on the other hand, introduced a continuum version of the model for which we proved that complete synchronization occurs for sufficiently high values of ε. Within the framework of our project this research was continued in several directions. In particular [54] extends the proof of ergodicity breaking from $N = 3$ to $N = 4$ sites. In [8], using spectral methods, the results on the continuum version are generalized from the case of the doubling map to a large class of piecewise expanding maps as individual dynamical systems at the sites. In another perspective, ergodicity breaking can be regarded as the emergence of multiple invariant measures in self-consistent dynamical systems, an example for which is discussed in [55].

1.4 Open dispersing billiards

An open dispersing billiard is obtained when removing only finitely many disjoint, smooth and strictly convex scatterers from the entire plane $\mathbb{R}^2$. It is a standing assumption that the configuration satisfies the non-eclipse condition: the convex hull of any two scatterers is disjoint from all remaining scatterers. This ensures that the periodic points have a natural coding associated to a subshift of finite type. The length of the periodic orbits together with their natural coding is called the marked length spectrum. We are mainly interested in the problem of spectral determination related to Mark Kac's famous question "Can one hear the shape of a drum?" ([41]) which in this dynamical context can be interpreted as follows: is it possible to reconstruct the geometry of the configuration from the marked length spectrum? The investigation of open dispersing billiards from this perspective was initiated in our paper [5], where in particular we proved that the marked length spectrum determines the marked Lyapunov spectrum. In the follow up work [29] the question of spectral determination is answered in the affirmative for open dispersing billiards with three scatterers subject to some symmetry and genericity conditions.

2 Fractals and Related Models

Let $\Phi = \{f_i\}_{i=1}^N$ be a finite set of contracting maps, mapping $\mathbb{R}^d$ into itself. Hutchinson [39] showed that there exists a unique non-empty compact set X, called the *attractor* of Φ, which satisfies $X = \bigcup_{i=1}^N f_i(X)$. Moreover, for every probability vector $(p_1, \dots, p_N)$ there exists a unique probability measure μ, called *stationary* measure, supported on X such that $\mu = \sum_{i=1}^N p_i (f_i)_* \mu$, where $f_* \mu = \mu \circ f^{-1}$ is the push forward of μ with respect to f.

2.1 Conformal systems

In the special case, when the maps are similarities, Hutchinson [39] gave a natural upper bound on the upper box-counting dimension of the attractor and the upper packing dimension of the stationary measure, called similarity- and Lyapunov dimension, respectively. Hochman's Theorem [35] states that under the exponential separation condition the Hausdorff dimension of the attractor equals the similarity dimension and the lower Hausdorff dimension of the stationary measure equals to the Lyapunov dimension. So far it was an open question whether the super-exponential condensation (i.e. no exponential separation) implies exact overlap between two different iterate of the maps. Bárány and Käenmäki [14] showed a counterexample, that is, there exists a self-similar IFS such that there is no exact overlap but the system is not exponentially separated.

We studied self-similar systems in further aspects. Bárány and Szvák [23] studied the dimension of stationary measures for systems with non-distinct fixed points, while Simon and Vágó [59] considered some parametrized families of IFSs formed by similarities (like orthogonal projections of Sierpiński-carpet) and showed that the set of parameters for which the stationary measure is singular forms a dense-G_δ set. Note that the result of Shmerkin and Solomyak [56] implies that the dimension of the directions that the projection of the natural stationary measure on the Sierpiński-carpet is singular has Hausdorff dimension 0 but the result of Simon and Vágó [59] implies that it has packing dimension 1.

In the more general setting when the maps are conformal, Bárány, Kolossváry, Rams and Simon [21] studied the Assouad dimension and the positivity of the Hausdorff measure of the attractor. Furthermore, Prokaj and Simon [52] studied systems, where the maps are only piecewise linear maps and determined for typical parameters the dimension of the attractor.

2.2 Non-conformal systems

The dimension theory becomes significantly more difficult if the maps of the IFS are affine maps. Falconer [30] generalized the concept of similarity dimension, called affinity dimension in this case, while Jordan, Pollicott and Simon [40] generalized the concept of the Lyapunov dimension for the affine setting. In both cases, these values are equal to the dimensions of the attractor and the stationary measure for almost all translation parameters if the norm of the matrices is at most 1/2.

Bárány, Hochman and Rapaport [11] proved a long stating conjecture, namely, for planar self-affine sets and measures if the IFS satisfies the strong open set condition and the matrices are strongly irreducible then the dimension of the attractor and the stationary measure equal to the affinity and the Lyapunov dimension respectively. This result has been applied later widely, like Bárány, Jordan, Käenmäki and Rams [12], who studied the Birkhoff-spectrum of continuous potentials and the Lyapunov-spectrum over planar self-affine sets, and like Bárány, Rams and Simon [20], who studied the dimension of repellers of piecewise affine expanding maps combining results of [11] with Hofbauer, Raith and Simon [36].

A natural step-forward in the understanding of the geometric properties of self-affine systems is to study its local structure and its proper-dimensional Hausdorff measure. Bárány, Käenmäki and Rossi [15] studied the Assouad dimension of self-affine sets under strong technical assumptions, which has been significantly generalized later by Bárány, Käenmäki and Yu [16]. Furthermore, in [16] the Hausdorff measure of planar self-affine sets was also studied under strong separation condition with dominated linear parts and the positivity of the Hausdorff measure was characterized explicitly.

Kolossváry and Simon [42] studied the dimension of reducible planar systems, more precisely, generalized Gatzouras-Lalley carpets with overlaps, for which [11] is not applicable.

Our knowledge becomes significantly limited in the most general case, namely, when the maps of the IFS are general C^1 non-conformal and non-linear mappings. In this case, even it was unknown so far whether the upper box-counting dimension of the attractor is bounded above by the singularity dimension, which is the natural generalisation of the affinity dimension, and whether the upper packing dimension of the stationary measures is at most the Lyapunov dimension. These were verified by Feng and Simon [32] recently. Furthermore, Feng and Simon [31] introduced a generalized transversality condition and showed in the cases when the derivative matrices are diagonal or lower-triangular with certain order of the modulus of the diagonal elements, that the dimension of the attractor and the stationary measure equal to the singularity- the Lyapunov-dimension respectively for Lebesgue almost every translation parameters if the norm of the derivative matrices is at most 1/2.

2.3 Multifractal analysis and dynamically defined subsets

Multifractal analysis plays an important role in the theory of fractal geometry. We have already studied the Birkhoff-spectrum in [12] over planar self-affine sets. Bárány, Rams and Shi [18] studied the topological entropy spectrum of weighted Birkhoff-averages, which are natural generalizations of the usual Birkhoff-average, of continuous potentials over subshifts of finite type, where the weights were chosen randomly, and established a conditional variational principle. Later, Bárány, Rams and Shi [19] studied the entropy and packing spectrum of weighted Birkhoff-averages with deterministic monotone decreasing positive weights and established its relation with the standard Birkhoff-spectrum.

Another important aspect of the multifractal analysis is the dimension of shrinking target subsets of the attractors of IFSs motivated by the Diophantine approximation. Allen and Bárány [1] studied the Hausdorff measure of shrinking targets of self-conformal sets with open set condition defined by geometric balls, Bárány and Rams [17] studied the Hausdorff dimension of shrinking targets of Bedford-McMullen carpets defined by geometric balls and by cylinder sets, while Bárány and Troscheit [24] studied the Hausdorff dimension of shrinking targets of generic self-affine sets defined by cylinder sets.

2.4 Chaos game

Barnsley [26] has introduced a natural Markov-chain on the attractor of an IFS $\{f_i\}_{i=1}^N$, called the chaos game, by using place dependent probability vectors $(p_1(x), \ldots, p_N(x))$ with transition probability $p_i(x)$ from x to $f_i(x)$. By using such Markov chain, it is possible to plot the attractor of an IFS relatively quickly. Bárány, Jurga and Kolossváry [13] determined the exact rate of the convergence of a typical orbit in Hausdorff metric to the attractor. The chaos game has a unique stationary measure which Hausdorff dimension and absolute continuity has been studied by Bárány, Simon, Solomyak and Śpiewak [22] for parametrized families of conformal IFS on the line.

2.5 Interactions between fractal geometry and Fourier analysis

The study of the algebraic sums has an important role both in dynamical systems and in geometric measure theory. Motivated by some configuration problems in geometric measure theory, Simon and Taylor [57] have investigated the algebraic sum $A + \Gamma$, where $A \subset \mathbb{R}^2$ and Γ is a piecewise C^2 curve. They determined the Hausdorff dimension and the positivity of the Lebesgue measure of $A + \Gamma$ with respect to the Hausdorff dimension of A. Furthermore, Simon and Taylor [58] showed an example such that the set $A + S^1$ has empty interior although A has full Lebesgue measure and it is a dense-G_δ set, but they also showed that if the set A with large measure and having general product like structure we have that the interior of $A+\Gamma$ is non-empty for C^2-curves Γ with non-vanishing curvature.

3 Fractal networks

Research on fractal networks is a dynamically growing field of network science. A central issue is to analyze the fractality of the networks. The fractal property can be identified using the box-covering algorithm. As this problem is known to be NP-hard, a plethora of approximating algorithms have been proposed throughout the years. Our study on the box-covering algorithms [44] established a unified framework for comparing approximating algorithms by collecting, implementing, and evaluating these methods in various aspects including running time and approximation ability.

In another work of ours on the fractality of networks, we introduced a new concept [43]: the transfinite fractal dimension of graph sequences that was motivated by the notion of fractality of complex networks proposed by Song et al. We showed how the definition of fractality can be modified to be able to apply it to networks with tree-like structure and exponential growth rate of neighborhoods. We also further generalized the concept of box dimension and introduced the transfinite Cesaro fractal dimension. Using rigorous proofs, we determined the optimal box-covering and transfinite fractal dimension of various models: the hierarchical graph sequence model, Song–Havlin–Makse model, spherically symmetric trees, and supercritical Galton–Watson trees.

3.1 Data-driven network analysis

Besides the rigorous mathematical way of investigating networks, we also study them using a data-driven approach. For example, in one of our recent works [51], we studied how well real networks can be described with a small selection of graph metrics, furthermore how well network models can capture the relations between graph metrics observed in real networks. This work unifies several branches of data-driven complex network analysis, such as the study of graph metrics and their pair-wise relationships, network similarity estimation, model calibration, and graph classification. We found that network domains can be efficiently determined using a small selection of metrics, moreover, we found that the models lack the capability of generating a graph with a high clustering coefficient and relatively large diameter simultaneously. On the other hand, models can capture exactly the degree-distribution-related metrics.

In another data-driven network analysis project, using bibliographic and co-authorship network analysis methods, we studied how the research area of network science evolved over the last twenty years [48]. We construct the co-authorship network of 56,646 network scientists and we analyzed its topology, dynamics, diversity, and interdisciplinarity. We also identified the most central authors, the largest communities, investigated the spatiotemporal changes, and compared the properties of the nodes to scientometric indicators.

Finally, in another related work we studied the robustness and error-tolerance of complex networks [49]. We considered a graph and we assigned colors to the vertices or edges, where the color-classes correspond to the shared vulnerabilities. An important problem is to find robustly connected vertex sets: nodes that remain connected to each other by paths providing any type of error. This is also known as color-avoiding percolation. We studied various possible modeling approaches of shared vulnerabilities, we analyzed the computational complexity of finding the robustly connected components.

3.2 Applications in education, social sciences, and telecommunication

In another line of research, we apply the modern tools of statistics and machine learning to answer research questions arising in various fields such as education, telecommunication, and social/behavioral sciences. We have several papers in the field of educational data science [37, 47, 50]. For example, in one of our works, we employed and evaluated several machine learning algorithms to identify students at-risk and predict student dropout of university based on the data available at the time of enrollment (secondary school performance, personal details) [50]. Moreover, we also presented a data-driven decision support platform for education directorate and stakeholders, and in addition, we provided an efficient visualization tool to analyze student flow patterns by alluvial and Sankey diagrams[37]. In another work, we applied the tools of network science, namely we introduced a data-driven probabilistic student flow approach to characterize prerequisite networks and study the distribution of graduation time based on the network topology and on the completion rate of the courses [47].

In another recent data science project of ours, we analyzed the data of 129,326 memes collected from Reddit in the middle of March, 2020, when the most serious coronavirus restrictions were being introduced around the world [25]. This work not only provides a looking glass into the thoughts of Internet users during the COVID-19 pandemic, but we also performed a content-based predictive analysis of what makes a meme go viral. Using machine learning methods, we also studied what incremental predictive power image related attributes have over textual attributes on meme popularity.

Finally, we have conducted research on anomaly detection in telecommunication data sets. We have developed a novel algorithm [38] that has a special feature: it not only indicates whether an observation is anomalous or not but also tells what exactly makes an anomalous observation unusual. Hence, it provides support to localize the reason of the anomaly. The proposed approach is model based; it relies on the multivariate probability distribution associated with the observations. Since the rare events are present in the tails of the probability distributions, we used copula functions, which are able to model the fat-tailed distributions well. The presented procedure scales well; it can cope with a large number of high-dimensional samples. Furthermore, our procedure can cope with missing values as well, which occur frequently in high-dimensional datasets.

A comment about the change of the participants of the project.

There have been no changes in the senior participants of the project. However, during the five years of the project some of our young participants joined to our group with the permission of the NKFI office. They were: Máté Baranyi , Said Abdelkhalek Fatma ,Gabrielly Keszthelyi, Vilma Orgoványi, Dániel Rudolf Prokaj.

References

[1] D. Allen and B. Bárány. On the hausdorff measure of shrinking target sets on self-conformal sets. *Mathematika*, 67(4):807–839, 2021.

[2] V. Baladi, M. F. Demers, and C. Liverani. Exponential decay of correlations for finite horizon Sinai billiard flows. *Invent. Math.*, 211(1):39–177, 2018.

[3] P. Bálint, H. Bruin, and D. Terhesiu. Periodic lorentz gas with small scatterers. *arXiv:2107.10529*, 2021.

[4] P. Bálint, O. Butterley, and I. Melbourne. Polynomial decay of correlations for flows, including Lorentz gas examples. *Comm. Math. Phys.*, 368(1):55–111, 2019.

[5] P. Bálint, J. De Simoi, V. Kaloshin, and M. Leguil. Marked length spectrum, homoclinic orbits and the geometry of open dispersing billiards. *Comm. Math. Phys.*, 374(3):1531–1575, 2020.

[6] P. Bálint, T. Gilbert, P. Nándori, D. Szász, and I. P. Tóth. On the limiting markov process of energy exchanges in a rarely interacting ball-piston gas. *Journal of Statistical Physics*, 166(3-4):903–925, 2017.

[7] P. Bálint, T. Gilbert, D. Szász, and I. P. Tóth. What mathematical billiards teach us about statistical physics? *Pure Appl. Funct. Anal.*, 6(1):1–35, 2021.

[8] P. Bálint, G. Keller, F. M. Sélley, and I. P. Tóth. Synchronization versus stability of the invariant distribution for a class of globally coupled maps. *Nonlinearity*, 31(8):3770–3793, 2018.

[9] P. Bálint and I. Melbourne. Statistical properties for flows with unbounded roof function, including the Lorenz attractor. *J. Stat. Phys.*, 172(4):1101–1126, 2018.

[10] P. Bálint, P. Nándori, D. Szász, and I. P. Tóth. Equidistribution for standard pairs in planar dispersing billiard flows. *Ann. Henri Poincaré*, 19(4):979–1042, 2018.

[11] B. Bárány, M. Hochman, and A. Rapaport. Hausdorff dimension of planar self-affine sets and measures. *Invent. Math.*, 216(3):601–659, 2019.

[12] B. Bárány, T. Jordan, A. Käenmäki, and M. Rams. Birkhoff and Lyapunov spectra on planar self-affine sets. *Int. Math. Res. Not. IMRN*, (10):7966–8005, 2021.

[13] B. Bárány, N. Jurga, and I. Kolossváry. On the Convergence Rate of the Chaos Game. *International Mathematics Research Notices*, 01 2022. rnab370.

[14] B. Bárány and A. Käenmäki. Super-exponential condensation without exact overlaps. *Adv. Math.*, 379:Paper No. 107549, 22, 2021.

[15] B. Bárány, A. Käenmäki, and E. Rossi. Assouad dimension of planar self-affine sets. *Trans. Amer. Math. Soc.*, 374(2):1297–1326, 2021.

[16] B. Bárány, A. Käenmäki, and H. Yu. Finer geometry of planar self-affine sets. *arXiv preprint arXiv:2107.00983*, 2021.

[17] B. Bárány and M. Rams. Shrinking targets on Bedford-McMullen carpets. *Proc. Lond. Math. Soc. (3)*, 117(5):951–995, 2018.

[18] B. Bárány, M. Rams, and R. Shi. On the multifractal spectrum of weighted Birkhoff averages. *Discrete Contin. Dyn. Syst.*, 42(5):2461–2497, 2022.

[19] B. Bárány, M. Rams, and R. Shi. Spectrum of weighted birkhoff average. *to appear in Studia Math.*, arXiv:2109.03898, 2022.

[20] B. Bárány, M. Rams, and K. Simon. Dimension of the repeller for a piecewise expanding affine map. *Ann. Acad. Sci. Fenn. Math.*, 45(2):1135–1169, 2020.

[21] B. Bárány, K. Simon, I. Kolossváry, and M. Rams. Hausdorff measure and Assouad dimension of generic self-conformal IFS on the line. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(6):2051–2081, 2021.

[22] B. Bárány, K. Simon, B. Solomyak, and A. Śpiewak. Typical absolute continuity for classes of dynamically defined measures. *Adv. Math.*, 399:Paper No. 108258, 73, 2022.

[23] B. Bárány and E. Szvák. On the dimension of self-similar measures with complicated overlaps. *Math. Nachr.*, 294(4):657–671, 2021.

[24] B. Bárány and S. Troscheit. Dynamically defined subsets of generic self-affine sets. *to appear in Nonlinearity*, arXiv:2109.06727, 2022.

[25] K. Barnes, T. Riesenmy, M. D. Trinh, E. Lleshi, N. Balogh, and R. Molontay. Dank or not? analyzing and predicting the popularity of memes on reddit. *Applied Network Science*, 6(1):1–24, 2021.

[26] M. Barnsley. *Fractals everywhere.* Academic Press, Inc., Boston, MA, 1988.

[27] N. Chernov and D. Dolgopyat. Brownian Brownian motion. I. *Mem. Amer. Math. Soc.*, 198(927):viii+193, 2009.

[28] N. Chernov and R. Markarian. *Chaotic billiards*, volume 127 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2006.

[29] J. De Simoi, V. Kaloshin, and M. Leguil. Marked length spectral determination of analytic chaotic billiards with axial symmetries. *arXiv preprint arXiv:1905.00890*, 2019.

[30] K. J. Falconer. The Hausdorff dimension of self-affine fractals. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 103, pages 339–350. Cambridge Univ Press, 1988.

[31] D.-J. Feng and K. Simon. Dimension estimates for C^1 iterated function systems and repellers. Part II. *Ergodic Theory and Dynamical Systems*, 10.1017/etds.2021.92:1–36, 2021.

[32] D.-J. Feng and K. Simon. Dimension estimates for C^1 iterated function systems and repellers. Part I. *Ergodic Theory and Dynamical Systems*, 10.1017/etds.2022.41:1–34, 2022.

[33] B. Fernandez. Breaking of ergodicity in expanding systems of globally coupled piecewise affine circle maps. *J. Stat. Phys.*, 154(4):999–1029, 2014.

[34] P. Gaspard and T. Gilbert. Heat conduction and Fourier's law in a class of many particle dispersing billiards. *New Journal of Physics*, 10:103004, 2008. `https://iopscience.iop.org/article/10.1088/1367-2630/10/10/103004`.

[35] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Ann. of Math. (2)*, 180(2):773–822, 2014.

[36] F. Hofbauer, P. Raith, and K. Simon. Hausdorff dimension for some hyperbolic attractors with overlaps and without finite Markov partition. *Ergodic Theory and Dynamical Systems*, 27(04):1143–1165, 2007.

[37] D. M. Horváth, R. Molontay, and M. Szabó. Visualizing student flows to track retention and graduation rates. In *2018 22nd International Conference Information Visualisation (IV)*, pages 338–343. IEEE, 2018.

[38] G. Horváth, E. Kovács, R. Molontay, and S. Nováczki. Copula-based anomaly scoring and localization for large-scale, high-dimensional continuous data. *ACM Transactions on Intelligent Systems and Technology (TIST)*, 11(3):1–26, 2020.

[39] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[40] T. Jordan, M. Pollicott, and K. Simon. Hausdorff dimension for randomly perturbed self affine attractors. *Communications in mathematical physics*, 270(2):519–544, 2007.

[41] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[42] I. Kolossváry and K. Simon. Triangular Gatzouras-Lalley-type planar carpets with overlaps. *Nonlinearity*, 32(9):3294–3341, 2019.

[43] J. Komjáthy, R. Molontay, and K. Simon. Transfinite fractal dimension of trees and hierarchical scale-free graphs. *Journal of Complex Networks*, 7(5):764–791, 2019.

[44] P. T. Kovács, M. Nagy, and R. Molontay. Comparative analysis of box-covering algorithms for fractal networks. *Applied Network Science*, 6(1):1–37, 2021.

[45] J. Marklof and A. Strömbergsson. The Boltzmann-Grad limit of the periodic Lorentz gas. *Ann. of Math. (2)*, 174(1):225–298, 2011.

[46] J. Marklof and B. Tóth. Superdiffusion in the periodic Lorentz gas. *Comm. Math. Phys.*, 347(3):933–981, 2016.

[47] R. Molontay, N. Horváth, J. Bergmann, D. Szekrényes, and M. Szabó. Characterizing curriculum prerequisite networks by a student flow approach. *IEEE Transactions on Learning Technologies*, 13(3):491–501, 2020.

[48] R. Molontay and M. Nagy. Twenty years of network science: A bibliographic and co-authorship network analysis. In *Big data and social media analytics*, pages 1–24. Springer, 2021.

[49] R. Molontay and K. Varga. On the complexity of color-avoiding site and bond percolation. In *International Conference on Current Trends in Theory and Practice of Informatics*, pages 354–367. Springer, 2019.

[50] M. Nagy and R. Molontay. Predicting dropout in higher education based on secondary school performance. In *2018 IEEE 22nd international conference on intelligent engineering systems (INES)*, pages 000389–000394. IEEE, 2018.

[51] M. Nagy and R. Molontay. Network classification-based structural analysis of real networks and their model-generated counterparts. *Network Science*, pages 1–24, 2022.

[52] R. D. Prokaj and K. Simon. Piecewise linear iterated function systems on the line of overlapping construction. *Nonlinearity*, 35(1):245–277, 2022.

[53] F. Sélley and P. Bálint. Mean-field coupling of identical expanding circle maps. *J. Stat. Phys.*, 164(4):858–889, 2016.

[54] F. M. Sélley. Symmetry breaking in a globally coupled map of four sites. *Discrete Contin. Dyn. Syst.*, 38(8):3707–3734, 2018.

[55] F. M. Sélley. A self-consistent dynamical system with multiple absolutely continuous invariant measures. *J. Comput. Dyn.*, 8(1):9–32, 2021.

[56] P. Shmerkin and B. Solomyak. Absolute continuity of self-similar measures, their projections and convolutions. *Trans. Amer. Math. Soc.*, 368(7):5125–5151, 2016.

[57] K. Simon and K. Taylor. Dimension and measure of sums of planar sets and curves. *arXiv preprint arXiv:1707.01407*, 2017.

[58] K. Simon and K. Taylor. Interior of sums of planar sets and curves. *Math. Proc. Cambridge Philos. Soc.*, 168(1):119–148, 2020.

[59] K. Simon and L. Vágó. Singularity versus exact overlaps for self-similar measures. *Proc. Amer. Math. Soc.*, 147(5):1971–1986, 2019.

[60] D. Szász. *Markov Approximations and Statistical Properties of Billiards*, pages 299–319. Springer International Publishing, Cham, 2019.

[61] D. Szász and T. Varjú. Limit laws and recurrence for the planar Lorentz process with infinite horizon. *J. Stat. Phys.*, 129(1):59–80, 2007.

[62] I. P. Tóth. Generalized Hölder continuity and oscillation functions. *Math. Phys. Anal. Geom.*, 21(4):Paper No. 34, 18, 2018.