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## 1. Toric ideals of flow polytopes

In our joint work with M. Domokos we studied toric ideals of low dimensional flow polytopes.

Let us denote by $Q$ a finite acyclic directed graph, in which multiple arrows are allowed, and let us denote by $Q_0$ and $Q_1$ the vertex and arrow sets of $Q$ respecitvely. For an integer *weight* $\theta \in \mathbb{Z}^{Q_0}$ consider the affine subspace:

$$\mathcal{A}(Q,\theta) = \{x \in \mathbb{R}^{Q_1} \mid \forall v \in Q_0 : \sum_{a^+=v} x(a) - \sum_{a^-=v} x(a) = \theta(v)\}$$

in $\mathbb{R}^{Q_1}$. Then the set

$$\nabla(Q,\theta) = \{x \in \mathcal{A}(Q,\theta) \mid \forall a \in Q_1 : x(a) \geq 0\}.$$

is a lattice polytope (i.e. a convex hull of points from $\mathbb{Z}^{Q_0}$) called a *flow polytope.* By standard constructions of toric geometry to a flow polytope $\nabla$ one associates a projective toric variety $X_\nabla$ whose vanishing ideal $\mathcal{I}_\nabla$ is generated by all the binomials

$$\prod_{z\in\nabla\cap\mathbb{Z}^d} t_z^{\alpha(z)} - \prod_{z\in\nabla\cap\mathbb{Z}^d} t_z^{\beta(z)},$$

where $\alpha,\beta \in \mathbb{N}_0^{\nabla\cap\mathbb{Z}^d}$ satisfy $\sum_{z\in\nabla\cap\mathbb{Z}^d} \alpha(z)z = \sum_{z\in\nabla\cap\mathbb{Z}^d} \beta(z)z \in \mathbb{Z}^d$ and $\sum_{z\in\nabla\cap\mathbb{Z}^d} \alpha(z) = \sum_{z\in\nabla\cap\mathbb{Z}^d} \beta(z) \in \mathbb{N}_0$.

The toric varieties associated to flow polytopes also arise as moduli spaces of so called *thin quiver representations*, which were studied in several papers (for example [1], [2], [13] and [14]) and were the original motivation of our research in this direction (see [8] for our earlier results). The notion also arises naturally in combinatorial optimization, in the study of integer network flows (see [25]). A notable subclass is the class of transportation polytopes, including the Birkhoff polytopes. Toric ideals of Birkhoff polytopes were applied in algebraic statistics in [7] and [6]. Our investigation focused on the following questions:

**Question 1.1.** (i) *Does $\mathcal{I}_\nabla$ have a cubic Gröbner basis for each flow polytope $\nabla$? If not, give a good degree bound for the generators of an appropriate initial ideal of $\mathcal{I}_\nabla$.*
(ii) *What are the flow polytopes $\nabla$ for which $\mathcal{I}_\nabla$ is quadratically generated?*
(iii) *What are the flow polytopes $\nabla$ for which $\mathcal{I}_\nabla$ has a quadratic Gröbner basis?*

The answer to these questions were known for 3 x 3 transportation polytopes that are not multiples of the Birkhoff polytope from [12]. It was also shown in [24] that the multiples of the Birkhoff polytope posssess a square-free quadratic initial ideal when the multiplier is even or divisble by 3. Generalizing these results, we considered *all* flow polytopes of dimension at most 4, and in [9, Corollary 7.1] we give an answer for Question 1.1, showing that up to dimension 4 a flow polytope always has a quadratic Gröbner basis unless it is equivalent to the Birkhoff polytope, which has a cubic Gröbner basis.

The key tool to find Gröbner basis of these toric ideals is to construct regular unimodular triangulations of the corresponding polytopes, as it is well known that the minimal non faces of these will be the leading terms of a Gröbner basis consisting of binomials. In order to achieve this goal we used methods from both [12] and results from our paper [8]. A flow polytope has a regular hyperplane subdivision into smaller flow polytopes, such that the minimal elements of this subdivision are compressed polytopes (i.e. polytopes with facet width 1). We gave a full classification of the compressed flow polytopes up to dimension 4 and constructed a pulling triangulation of each. Using the fact that these pulling triangulations glue together into a regular unimodular triangulation of the subdivided polytope we proved our main result [9, Theorem 6.4]:

**Theorem 1.2.** *Let $\nabla$ be a flow polytope of dimension $1 \leq \dim(\nabla) \leq 4$. If $\nabla$ is not equivalent to the Birkhoff polytope then it has a quadratic triangulation. The Birkhoff polytope has a regular unimodular triangulation whose minimal non-faces have at most 3 elements.*

We note that this also gives sharper results in the context of [12] and [24], as the broader class of flow polytopes allows more effective proofs by induction.

Constructing triangulations also play in important role in enumerative combinatorics, in particular in calculating Ehrhart polynomials. Ehrhart polynomials of certain flow polytopes were intensively studied in recent years (see for example [3], [22]), motivated partly by the observation that in certain cases they count the dimension of weight spaces of representations of compact Lie groups. We used our results to compute the Ehrhart polynomials of the prime compressed 3 and 4-dimensional flow polytopes (see [9, Proposition 9.1]).

## 2. Tropical prime ideals

In our joint work with K. Mincheva we studied prime ideals of tropical polynomial semirings, and gave a full classification of these, including a description of the tropical ideals among them which are the central objects on the algebraic side of tropical geometry.

Tropical geometry provides a new set of purely combinatorial tools, which has been used to approach classical problems. Some applications include computing Gromov-Witten invariants due to Mikhalkin in [23], tropical proof of the Brill-Noether Theorem [4], Brill-Noether theory for curves of a fixed gonality [15], developing a strategy to attack the Riemann hypothesis [5], the Gross-Siebert program in mirror symmetry [11], and studying toric degenerations [18].

In tropical geometry most algebraic computations are done on the classical side by using the algebra of the original variety. The theory developed so far has explored the geometric aspect of tropical varieties as opposed to the underlying (semiring) algebra. There are still many commutative algebra tools and notions without a tropical analogue. In the recent years, there has been a lot of effort dedicated to developing the necessary tools for commutative algebra using different frameworks, among which prime congruences ([16], [17]), tropical ideals ([19], [20]), tropical schemes introduced in [10]. These approaches allow for the exploration of the properties of tropicalized spaces without tying them up to the original varieties and working with geometric structures inherently defined in characteristic one (that is, over additively idempotent) semifields. In our current research we were mostly interested in the case when the underlying semifield is the tropical semifield $\mathbb{T}$. It is the set $\{\mathbb{R} \cup \{-\infty\}\}$ with two operations: maximum playing the role of addition and addition acting as multiplication.

In additively idempotent semirings, there is no bijection between ideals and congruences, i.e. equivalence relations that respect the operations. However there is a central construction in tropical geometry that associates to each ideal in the polynomial semiring $\mathbb{T}[x_1, \ldots, x_n]$ a congruence relation, called the *bend congruence* of the ideal. To recall the definition let us write $supp(f)$ for the support of a polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ and for $a \in supp(f)$ let us denote by $f_{\hat{a}}$ the polynomial we obtain by deleting $a$ from $f$. Then the *bend relations of* $f$ is the set of relations

$$bend(f) = \{(f \sim f_{\hat{a}})\}_{a \in supp(f)}.$$

The *bend congruence* of an ideal $I$, denoted $Bend(I)$, is the congruence generated by the bend relations of all the elements of $I$. Conversely, to a congruence $C$, one can associate the ideal $I_C$ of all polynomials, whose bend relations are contained in the congruence. If $I_{Bend(I)} = I$ then we say that the ideal $I$ is *bend-closed*.

By the *vanishing locus* $V(C)$ of a congruence $C$ of $\mathbb{T}[x_1, \ldots, x_n]$ we just mean the set of points in $\mathbb{T}^n$ where all the relations of $C$ are satisified, as in the points $x \in \mathbb{T}^n$ where $f_1(x) = f_2(x)$ whenever $f_1 \sim_C f_2$. The vanishing locus $V(I)$ of an ideal $I$ is $V(Bend(I))$.

Tropical ideals are ideals of $\mathbb{T}[x_1, \ldots, x_n]$, which satisfy that the polynomials of degree at most $d$ are the vectors of a valuated matroid. Tropical ideals fully capture the algebra of tropical varieties, which are realized as the vanishing locus of an ideal of tropical polynomials. For a more precise definition we refer the reader to [20] where these ideals were first introduced. Tropical ideals are called realizable whenever they arise as the image of classical ideals under a valuation map. However, not every tropical ideal comes as the tropicalization of some classical object. We note that classical introductions to tropical geometry only consider the realizable tropical ideals and their vanishing loci, however from an algebraic perspective it is more convenient to work in the more general framework of [20].

The primary goal of our research during the grant period was to gain a better understanding of the prime ideals in $\mathbb{T}[x_1, \ldots, x_n]$ and to identify the tropical ideals amongst them as they are key to establishing the desired algebro-geometric framework in this context, and previously no classification of either of these sets were known. Furthermore we wanted to explore if a good notion of dimension can be established using these ideals, that matches the intuitive dimension of the corresponding tropical varieties. Finally we were interested if Nullstellensatz type results can be obtained for the class of bend-closed or tropical ideals.

Our key results were obtained using a tool that we developed in our earlier work in [16]. There a notion of *prime congruences* were introduced for commutative semirings, which in additively idempotent semirings is equivalent to quotient semiring of the congruence being cancellative and totally ordered under the ordering coming from the idempotent addition. Our first result showed that the elements of this class are in bijection with the bend-closed prime ideals.

**Theorem 2.1.** *Let $I$ be a bend-closed ideal of $\mathbb{T}[x_1, \ldots, x_n]$, then $I$ is prime if and only if $Bend(I)$ is a prime congruence.*

As in [16] a complete description of the prime congruences of $\mathbb{T}[x_1, \ldots, x_n]$ was given in terms of monomial orderings, the above gives us a full understanding of the bend-closed prime ideals. An important corollary of this result is the following:

**Theorem 2.2.** *The vanishing locus of the bend relations of any prime ideal in $\mathbb{T}[x_1, \ldots, x_n]$ is at most one point.*

Next we applied this result to derive a characterization of prime tropical ideals:

**Theorem 2.3.** *The prime tropical ideals in $\mathbb{T}[x_1, \ldots, x_n]$ are in bijection with the points of $\mathbb{T}^n$. The prime ideal corresponding to the point $x \in \mathbb{T}^n$ consists of all polynomials that take their maximum twice at $x$.*

This result implies that there are no chains of tropical ideals longer than 1, hence one can not hope that they yield a useful notion of dimension. Instead we turned to results from our earlier work in [17] where Krull dimension of congruences was defined as the maximal length of a chain of prime congruences (in the sense of [16]) containing the congruence (this notion coincides with the usual Krull dimension when the semiring is a ring). We proved in our current work that this definition gives us the dimension of the vanishing locus of tropical ideals, and applying Theorem 2.2 obtained the following result, that shows that the bend-closed ideals induce a good notion of Krull dimension.

**Theorem 2.4.** *Let $I$ be a tropical ideal in $\mathbb{T}[x_1, \ldots, x_n]$ such that $X = V(I)$ is of dimension $d$. Then the length of the longest chain of bend-closed prime ideals containing $I$ is $d + 1$.*

It was left to classify the rest of the prime ideals of $\mathbb{T}[x_1, \ldots, x_n]$. One can find a wealth of examples that are somewhat counter intuitive coming from a ring theoretical perspective. For example it is straightforward to check that in $\mathbb{T}[x, y]$ the ideal that consists of all polynomials that are inhomogeneous in $y$ is a prime ideal that is not bend-closed. It turned out that all prime ideals arise in a somewhat similar fashion. For prime congruences $P_1, P_2$ let us denote by $I(P_1, P_2)$ the ideal with non-zero elements the polynomials whose leading term with respect to $P_1$ has monomials from at least two distinct equivalence class of $P_2$. Note that depending on the choice of $P_1, P_2$, the ideal $I(P_1, P_2)$ might be the same as the bend-closed ideal $I_{P_1}$. We proved the following result:

**Theorem 2.5.** *For any prime congruences $P_1, P_2$ of $\mathbb{T}[x_1, \ldots, x_n]$ the ideal $I(P_1, P_2)$ is prime and the intersection of the bend-closed ideals containing $I(P_1, P_2)$ is $I_{P_1}$. Moreover every prime congruence of $\mathbb{T}[x_1, \ldots, x_n]$ arises this way.*

Our last research goal was to see if a Nullstellensatz type result can be derived from the above classification, in particular to see that for a polynomial $f$ and an ideal $I$ the bend relations of $f$ hold on $V(I)$ if and only if $f$ lies in the radical of $I$ (i.e. some power of $f$ is in $I$). It was conjectured by MacLagan and Rincón ([20]) that this holds for tropical ideals, and the present authors conjectured that it might hold for the wider class of bend-closed ideals. One can conclude by the usual ring theoretical argument that for an ideal $I$ in $\mathbb{T}[x_1, \ldots, x_n]$ the radical of $I$ is the same as the intersection of all the primes containing $I$. To prove a Nullstellensatz for some class of ideals one would have to show that the intersection of all primes lying above the ideals in question is the same as the intersection of the bend-closed primes. We managed to construct bend-closed counterexamples to this conjecture. So far we did not settle the conjecture in the tropical case, but based on our investigations in this direction, we suspect that a counterexample might exist.

The results above have been presented in several talks and are part of a paper in preparation which will soon appear on the arxiv. Our earlier results on congruences have appeared in [16] and [17].

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