RESEARCH REPORT
Random graphs and correlated percolation models
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In this document I describe the results of my research for the three years that I have spent as a postdoctoral fellow of NKFI from 2016 to 2019 at the Department of Stochastics of the Mathematics Institute of Budapest University of Technology and Economics (BME).

1 Random graphs

1.1 Age evolution in the mean field forest fire model via multitype branching processes

This joint work (42 pages) with Edward Crane (Bristol) and Dominic Yeo (Oxford) is submitted to Annals of Probability. The decision of editors after the first round of referee reports was "Major revision required", and currently we are addressing the requests of the referees. The version submitted to AoP is on arXiv and it is also linked in the electronic list of publications of this NKFI PD final report.

The *mean field forest fire (MFFF)* process was introduced by Ráth and Tóth in [10]. It can be viewed as an adjustment to the Erdős–Rényi dynamics, which destroys the edges of potential giant components as they are forming, and thus maintains the system in a critical state forever.

- The model has n vertices, with some (possibly random) initial set of undirected edges at time 0.
- Each possible edge joining two vertices appears at rate $1/n$, independently.
- At rate $\lambda = \lambda(n)$ each vertex is struck by lightning, independently. When a vertex is struck by lightning, the vertex survives but all of the edges in its connected component (or cluster) are instantaneously deleted. Those edges may subsequently reappear.

The most interesting asymptotic regime for the lightning rate is $1/n \ll \lambda(n) \ll 1$. In this regime, clusters of any fixed finite size are destroyed at a negligible rate when n is large. However, the total rate of lightning strikes in the model diverges, so if a cluster of size comparable to n were able to form then it would only survive for time $o(1)$. This is the asymptotic regime of lightning rates for which the model displays self-organized criticality.

The goal of our paper is to describe the local graph structure of the MFFF at time t as $n \to \infty$ in terms of a multitype branching process, and also to give a simple description of the time evolution of the parameters that govern this multitype branching processes.

At any time $t \geq 0$ each vertex v has an *age* $a_t^n(v)$, which is defined to be the time since it was last burned, or $a_0^n(v) + t$ if it has not yet been burned. Let $\pi_t^n = \frac{1}{n}\sum_v \delta_{a_t^n(v)}$ be the empirical measure of these ages.

Our central observation is that conditional on the ages $a_t^n(v)$ and $a_t^n(w)$ of two vertices v and w, the probability that they are joined by an edge at time t is exactly $1-\exp(-a_t^n(v) \wedge a_t^n(w)/n)$. Furthermore, these events are independent for distinct pairs of vertices. So conditional on π_t^n, the graph seen at time t is an inhomogeneous random graph *(IRG)* in the sense of Bollobás et al [4].

We show that the empirical age distributions π_t^n converge as $n \to \infty$ to a deterministic limit distribution π_t, moreover $(\pi_t,\, t \geq 0)$ satisfies an autonomous differential equation, which we call the *age differential equation.* To describe this, we first recall from [10] that there exists a so-called *gelation time* $t_{\text{gel}} \geq 0$, at which the model makes a phase transition from subcritical to critical behaviour. For $0 \leq t < t_{\text{gel}}$ the age of each vertex simply increases at rate 1 unless it burns before t_{gel}. Only an asymptotically negligible proportion of the vertices burn before t_{gel}, so the limiting age distribution satisfies the simple transport equation

$$\frac{\mathrm{d}\pi_t}{\mathrm{d}t} = -\delta_0' * \pi_t\,. \tag{1.1}$$

Here δ_0' is the derivative of the Dirac delta at 0, so this statement is an equality of Schwarz distributions. The situation is more interesting for $t \geq t_{\text{gel}}$, when the model is critical. Then, for each such t, there exists a unique non-negative, continuous and non-decreasing function $s \mapsto \theta_t(s)$ satisfying $\int \theta_t(s)\, d\pi_t(s) = 1$ and

$$\theta_t(s) = \int_0^\infty \theta_t(u)\,(u \wedge s)\,\mathrm{d}\pi_t(u), \quad s \in [0, \infty]. \tag{1.2}$$

In fact, θ_t is the normalized right eigenfunction corresponding to the eigenvalue $\lambda_t = 1$ of the Perron-Frobenius operator associated with the (critical) multi-type branching process that arises as the local limit of the graph at time t as $n \to \infty$.

We define μ_t to be the probability measure absolutely continuous with respect to π_t with Radon–Nikodym derivative $\frac{\mathrm{d}\mu_t}{\mathrm{d}\pi_t}(s) = \theta_t(s)$. Then for $t > t_{\text{gel}}$, π_t satisfies the following distribution-valued differential equation:

$$\frac{\mathrm{d}\pi_t}{\mathrm{d}t} = -\delta_0' * \pi_t - \varphi(t)\mu_t + \varphi(t)\delta_0\,, \qquad \varphi(t) = \left(\int \theta_t(s)^3\,\mathrm{d}\pi_t(s)\right)^{-1}. \tag{1.3}$$

My conference and seminar talks about this result:

- 2019 March 18.

  Venue: Institute of Information Theory and Automation, Prague, Czech Republic

  Title: "On random graphs and forest fires"

- 2019 June 26.

  Venue: Felix-Klein Colloquium, University of Leipzig, Germany

  Title: "On random graphs and forest fires"

1.2 A moment-generating formula for Erdős-Rényi component sizes

This paper is published in Electronic Communications in Probability and it is also linked in the electronic list of publications of this NKFI PD final report.

The Erdős-Rényi graph $\mathcal{G}_{n,p}$ is the random graph on n vertices where each pair of vertices is connected with probability p, independently from each other. We denote by $\mathbb{P}_{n,p}$ the law of $\mathcal{G}_{n,p}$ and $\mathbb{E}_{n,p}$ the corresponding expectation. We assume that the vertex set of $\mathcal{G}_{n,p}$ is $[n] = \{1, \ldots, n\}$ and we denote by $\mathcal{C}$ the connected component in $\mathcal{G}_{n,p}$ of the vertex indexed by 1. We denote by $|\mathcal{C}|$ the number of vertices of $\mathcal{C}$.

For any $n \in \mathbb{N}$, $p \in [0, 1]$, $j \in \mathbb{Z} \cap (-n, +\infty)$, and $k \in [n]$ we define

$$g_{n,p}(j,k) = (1-p)^{jk} \prod_{i=0}^{k-1} \frac{n-i+j}{n-i}. \tag{1.4}$$

The central result of our paper is the following formula: for any $n \in \mathbb{N}$, $j \in \mathbb{Z} \cap (-n, +\infty)$ and $p \in [0,1]$ we have

$$\mathbb{E}_{n,p}\left[\, g_{n,p}(j, |\mathcal{C}|)\,\right] = \frac{n+j}{n}\left(1 - \mathbb{P}_{n+j,p}[\, |\mathcal{C}| > n\,]\right). \tag{1.5}$$

Note that if $j \leq 0$ then the r.h.s. is simply $\frac{n+j}{n}$, moreover the formulas (1.5) for $j \in \mathbb{Z} \cap (-n, 0]$ together uniquely characterize the distribution of $|\mathcal{C}|$ under $\mathbb{P}_{n,p}$.

This formula allows us to give elementary proofs of some results of [6] and [7] about the susceptibility in the subcritical graph and the CLT [9] for the size of the giant component in the supercritical graph.

My conference and seminar talks about this result:

- 2017 September 11.

  Venue: Randomness and Graphs : Processes and Structures, Eurandom, Eindhoven, The Netherlands

  Title: A moment-generating formula for Erdős-Rényi component sizes

1.3 The window process of slightly subcritical frozen percolation

This paper, joint work with Dominic Yeo (Oxford), is still in being typed. In 2018 April we were already confident enough to give a conference talk about the results. However, the typing was delayed, because we felt that other, more urgent projects had to be finished before this one. The current (unfinished) version is linked in the electronic list of publications of this NKFI PD final report.

The *mean field forest frozen percolation* process was introduced by Ráth in [11]. Similarly to the mean field forest fire model, it can be viewed as an adjustment to the Erdős–Rényi dynamics, which destroys large connected components.

- Initially, the model has n vertices and no edges.
- Each possible edge joining two vertices which are still alive at time t appears at rate $1/n$, independently.
- At rate λ_n each vertex which is still alive at time t is struck by lightning, independently. When a vertex is struck by lightning, its connected component (in the graph spanned by the set of alive vertices at time t) is instantaneously deleted (i.e., its vertices are removed from the set of alive vertices).

We assume $\ln(n)^{4/3} n^{-1/3} \ll \lambda_n \ll \frac{1}{\ln(n)}$. In this asymptotic regime of lightning rates, the model displays self-organized criticality, but the lightnings are frequent enough so that components of size $n^{2/3}$ never appear (this would be the order of magnitude of the size of the largest connected component in the critical window of the Erdős–Rényi graph), thus our model remains "slightly subcritical".

Let us denote by $A_n(t)$ the number of alive vertices at time t, thus $A_n(0) = n$. Denote by $(\mathcal{F}_n(t))$ the filtration generated by the process $(A_n(t))$. The starting point of our paper is the observation that given $\mathcal{F}_n(t)$, the graph spanned by the vertices which are alive at time t is distributed as an Erdős–Rényi graph with $A_n(t)$ vertices and edge probability $1 - e^{-t/n}$. As a consequence, $(A_n(t))$ is a Markov process (albeit not time-homogeneous).

We define the *window process* $(W_n(t))_{t \geq 0}$ by

$$W_n(t) = \frac{t A_n(t)}{n}. \tag{1.6}$$

Heuristically, given $\mathcal{F}_n(t)$, $W_n(t)$ is roughly equal to the expected number of neighbours of a vertex in the graph at time t. Thus, heuristically, if $W_n(t) > 1$ then the graph is "supercritical", if $W_n(t) < 1$ then it is "subcritical" and if $W_n(t) \approx 1$ then it is "critical". It was already proved in [11] that $W_n(t)$ converges in probability to $1 \wedge t$ as $n \to \infty$, thus for all $t \geq 1$ the graph is critical. In order to get the precise asymptotics of $W_n(t)$, let us denote by $a_n(t)$ the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} a_n(t) = -\lambda_n \cdot \frac{a_n(t)}{1 - t \cdot a_n(t)}, \qquad a_n(0) = 1. \tag{1.7}$$

Denote by

$$w_n(t) := a_n(t) \cdot t, \qquad t \geq 0. \tag{1.8}$$

Let us first note that for each t, we have $w_n(t) \to 1 \wedge t$ as $n \to \infty$. We will see that $w_n(t)$ is a good deterministic approximation of $W_n(t)$. We define the rescaled *window fluctuation process* as

$$Z_n(t) = (n\lambda_n)^{1/2} \cdot (W_n(t) - w_n(t)). \tag{1.9}$$

Our first main result describes the scaling limit of the window fluctuation process:

Theorem 1.1. *Let $t \in (1, +\infty)$. Let*

$$\widetilde{Z}_n(s) := Z_n(t + \lambda_n s) \overset{(1.9)}{=} (n\lambda_n)^{1/2} \cdot (W_n(t + \lambda_n s) - w_n(t + \lambda_n s)), \quad s \in \mathbb{R}. \tag{1.10}$$

Then

$$\widetilde{Z}_n(\cdot) \Rightarrow \widetilde{Z}(\cdot)$$

in $\mathbb{D}[-\widetilde{T}, \widetilde{T}]$ for all $\widetilde{T} < \infty$, where $\widetilde{Z}$ is the stationary Ornstein–Uhlenbeck process which satisfies the SDE

$$\mathrm{d}\widetilde{Z}(s) = -\frac{1}{t^2} \widetilde{Z}(s)\,\mathrm{d}s + \frac{1}{t}\,\mathrm{d}B(s). \tag{1.11}$$

Thus, heuristically, if $W_n(\cdot)$ gets too far from $w_n(\cdot)$, its drift pulls it back closer to $w_n(\cdot)$.

Our second main result confirms the "slightly subcritical" part of [11, Conjecture 1.1] by showing that the typical size of a burnt component is λ_n^{-2}:

Theorem 1.2. *Let $t \in (1, +\infty)$ and*

$$U_n(s) := n\lambda_n^2 \left(W_n \left(t + \frac{s}{n\lambda_n^2} \right) - W_n(t) \right). \tag{1.12}$$

Let $U(\cdot)$ be the Lévy process defined by

$$U(s) := \frac{1}{t} s - \frac{1}{t} \int_0^s \int_0^\infty y N(\mathrm{d}y, \mathrm{d}u), \tag{1.13}$$

where $N(\cdot, \cdot)$ is a Poisson point process on $\mathbb{R}_+^2$ with intensity $\frac{1}{\sqrt{2\pi}} y^{-3/2} e^{-y/2} \mathrm{d}y\mathrm{d}u$.

Then

$$U_n(\cdot) \Rightarrow U(\cdot)$$

in $\mathbb{D}[0, \widehat{T}]$ for each $\widehat{T} < \infty$.

Note that the limiting process $U(\cdot)$ is a pure jump process with constant upward drift and downward jumps, with a dense set of downward jump times.

The proof of the above stated results consists of multiple ingredients: (i) the analysis of the non-trivial asymptotics (as $\lambda_n \to 0$) of the ODE (1.7), (ii) comparison of the infinitesimal generators of the Markov processes $W_n(\cdot)$, $Z(\cdot)$ and $U(\cdot)$, which boils down to the study of the component size distribution of the slightly subcritical Erdős–Rényi graph (using the formula (1.5), among other tools).

My conference and seminar talks about this result:

- 2018 April 12.

  Venue: UK Easter Probability Meeting 2018, University of Sheffield, UK.

  Title: The window process of slightly subcritical frozen percolation

2 Correlated percolation models

2.1 On the threshold of spread-out voter model percolation

Joint work with Daniel Valesin (Groningen). This paper is published in Electronic Communications in Probability, and it is also linked in the electronic list of publications of this NKFI PD final report.

The *voter model* on $\mathbb{Z}^d$ with range $R \in \mathbb{N}$ is a Markov process $(\xi_t)_{t\geq 0}$ on $\{0,1\}^{\mathbb{Z}^d}$. In the usual interpretation, sites of $\mathbb{Z}^d$ represent individuals ("voters") and the states 0 and 1 represent two conflicting opinions. Individuals are all endowed with independent exponential clocks (all with parameter 1); whenever the clock of individual x rings, another individual y is chosen uniformly at random within ℓ^1-distance at most R from x, and then x copies the opinion of y.

Let $\mathcal{I}_{d,R}$ denote the set of extremal stationary distributions of the voter model on $\mathbb{Z}^d$ and range R. In case $d = 1$ or 2, for any R, this set consists only of $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$, the two measures that give full mass to the configurations which are identically equal to 0 or 1, respectively. In case $d \geq 3$, $\mathcal{I}_{d,R}$ consists of a one-parameter family of measures

$$\{\mu_{\alpha,R} : \ 0 \leq \alpha \leq 1\}.$$

Each of the measures $\mu_{\alpha,R}$ is invariant and ergodic with respect to translations in $\mathbb{Z}^d$. Additionally,

$$\mu_{\alpha,R}(\{\xi : \xi(0) = 1\}) = \alpha,$$

so that α is a density parameter.

For given values of d, R and α, let $\xi \in \{0,1\}^{\mathbb{Z}^d}$ be a configuration sampled from $\mu_{\alpha,R}$. Consider the subgraph of the nearest-neighbor lattice $\mathbb{Z}^d$ induced by the set of vertices $\{x : \xi(x) = 1\}$ (i.e., the set of *open* sites). Let Perc be the event that this subgraph contains an infinite connected component (*cluster*).

By ergodicity, $\mu_{\alpha,R}(\text{Perc})$ is either 0 or 1. The statement that the measures $\mu_{\alpha,R}$ exhibit a non-trivial percolation phase transition with respect to the density parameter α means that, for any $d \geq 3$ and $R \in \mathbb{N}$, there exists $\alpha_c = \alpha_c(R) \in (0,1)$ (depending on d and R) such that $\mu_{\alpha,R}(\text{Perc}) = 0$ if $\alpha < \alpha_c$ and $\mu_{\alpha,R}(\text{Perc}) = 1$ is $\alpha > \alpha_c$. It is already known (c.f. [13]) that this is indeed the case under two sets of assumptions: first, $d \geq 5$, and second, $d = 3$ or 4 and R large enough.

The main result of our paper is as follows. For any $d \geq 3$, as $R \to \infty$, the critical density value for percolation phase transition of the stationary measures of the voter model with range R converges to the critical density value for independent Bernoulli percolation:

$$\lim_{R\to\infty} \alpha_c(R) = p_c. \tag{2.1}$$

This convergence result seems natural, but the proof is not at all automatic, since it is known that $\mu_{\alpha,R}$ cannot be stochastically dominated (or minorated) by a Bernoulli product measure.

Our main tools are (i) the multi-scale renormalization methods of [12] and (ii) a new lemma, which allows for a direct comparison between the measures $\mu_{\alpha,R}$ and Bernoulli product measures, which relies on a natural coupling of systems of independent random walks, coalescing random walks (that are used in the construction of $\mu_{\alpha,R}$) and annihilating random walks.

My conference and seminar talks about this result:

- 2018 June 13.

  Venue: The 40th Conference on Stochastic Processes and their Applications, Gothenburg, Sweden.

  Title: On The Threshold Of Spread-out Voter Model Percolation

2.2 On the threshold of spread-out contact process percolation

This joint paper with Daniel Valesin (Groningen) is still being typed (currently 48 pages). We consider the paper to be nearly finished: the proofs of some minor lemmas need to be typed and the introduction needs to be written. The current version is linked in the electronic list of publications of this NKFI PD final report.

In the R-spread out d-dimensional contact process, each site of $\mathbb{Z}^d$ can be in state 0 (healthy) or 1 (infected). An infected site heals at rate one and at rate λ it infects a uniformly chosen vertex within a ball of radius R. It is known that there exists a critical threshold $\lambda_c(R)$ such that the upper stationary measure $\mu_{\lambda,R}$ of this interacting particle system is non-trivial if and only if $\lambda > \lambda_c(R)$. It follows from [5] that $\lambda_c(R)$ goes to 1 as R goes to infinity.

Similarly to the case of the voter model (discussed in the previous subsection), one may view a configuration sampled according to the upper stationary measure as a correlated nearest neighbour site percolation model and define the percolation threshold $\lambda_p(R)$ such that $\mu_{\lambda,R}(\text{Perc}) = 0$ if $\lambda < \lambda_p(R)$ and $\mu_{\lambda,R}(\text{Perc}) = 1$ is $\lambda > \lambda_p(R)$.

We prove that for any $d \geq 2$ we have

$$\lim_{R\to\infty} \lambda_p(R) = \frac{1}{1-p_c}, \tag{2.2}$$

where $p_c = p_c(d)$ is the critical percolation threshold of Bernoulli site percolation on $\mathbb{Z}^d$. This implies in particular that $\lambda_c(R) < \lambda_p(R)$ for large enough R, answering an open question of [8] in the R-spread-out case.

Our proof combines multi-scale renormalization, a new variant of the stochastic domination results of [8] and a coupling of a family of independent branching random walks with the graphical construction of the contact process.

My conference and seminar talks about this result:

- 2019 August 22.

  Venue: Workshop on Complex Systems, Institute of Information Theory and Automation, Prague, Czech Republic

  Title: On the threshold of spread-out contact process percolation

2.3 Frozen percolation on the binary tree is nonendogenous

Joint work with Jan M. Swart (Prague) and Tamás Terpai (ELTE), submitted recently (45 pages). The arXiv version is linked in the electronic list of publications of this NKFI PD final report.

Let (T, E) be a regular tree where each vertex has degree 3, and let $\mathcal{U} = (U_e)_{e\in E}$ be an i.i.d. collection of uniformly distributed $[0, 1]$-valued random variables, indexed by the edges of the tree. We write $E_t := \{e \in E : U_e \leq t\}$ ($t \in [0, 1]$). Aldous [1] has proved the following theorem.

Theorem 2.1. *It is possible to couple $\mathcal{U}$ to a random subset $F \subseteq E$ with the following properties:*

1. *$e \notin F$ if and only if no endvertex of e is part of an infinite cluster of $E_{U_e} \setminus (F \cup \{e\})$.*

2. *The law of $(\mathcal{U}, F)$ is invariant under automorphisms of the tree.*

At time $t \in [0,1]$, we call edges in $E_t \setminus F$ *open*, edges in $E_t \cap F$ *frozen*, and all other edges *closed*. Then property (i) can be described in word as follows. Initially all edges are closed. At time U_e, the edge e opens provided neither of its endvertices is at the time part of an infinite open cluster; in the opposite case, it freezes.

In [1, Section 5.7], Aldous asked whether the set F of frozen edges is measurable w.r.t. the σ-field generated by $\mathcal{U}$, and cautiously conjectured that this might indeed be the case. In [2, Thm 55], an apparent proof of this conjecture by Bandyopadhyay was announced that appeared on the arXiv [3] but turned out to contain an error. In the last posted update of [3] from 2006, Bandyopadhyay reported on numerical similations that suggested nonuniqueness, and from this moment on this seems to have been the generally held belief. Our main result is that almost sure uniqueness does not hold.

Theorem 2.2. *There exists a triple $(\mathcal{U}, F, F')$ such that $\mathcal{U} = (U_e)_{e\in E}$ is an i.i.d. collection of uniformly distributed $[0,1]$-valued random variables, F and F' are random subsets of E satisfying property (i) of Theorem 2.1, the law of $(\mathcal{U}, F, F')$ is invariant under automorphisms of the tree, and $F \neq F'$ a.s.*

The construction of Theorem 2.1 uses a so-called *recursive tree process* (RTP), c.f. [2, Section 2.3], and we prove Theorem 2.2 by showing that this RTP is *nonendogenous*, c.f. [2, Section 2.4]. In fact, we prove Theorem 2.2 by explicitly constructing a non-diagonal fixed point of the bivariate *recursive distributional equation* (RDE) associated to the frozen percolation RTP, c.f. [2, Theorem 11]. An essential role in our proofs is played by a frozen percolation process on a continuous-time binary Galton Watson tree that has nice scale invariant properties.

My conference and seminar talks about this result:

- 2019 September 20.

  Venue: Large Scale Stochastic Dynamics, Oberwolfach Workshop, Germany

  Title: Frozen percolation on the binary tree is nonendogenous

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