Final report on the OTKA grant no. K120186
Qualitative Properties of Delay Differential and Difference Equations

Our research is related to the study of the qualitative properties of differential and difference equations, mainly to stability, boundedness and positivity problems, the asymptotic characterization of solutions, the topic of integral and discrete inequalities and state-dependent delay equations. In the 2017-2021 research period, 30 publications were published in the framework of the present project, including 1 monograph and 27 journal articles with an impact factor (3 are published in a journal with rank D1, 8 with rank Q1 and 13 with rank Q2). The total impact factor was 43.701. In addition to the above publications, 10 additional papers were published in the framework of the parallel grant KH130513 by the same research group. We have registered 907 independent citations to our publications in the last 5 years, including 73 independent citations to our 30 papers of this grant. In 2018, we organized an international conference in Veszprém.

We give our short summaries of the publications grouped according to the areas defined in the grant application (topics 1-3 below). At the same time, papers related to broader areas are listed in topic 4 below.

1. Asymptotic behavior, stability, boundedness and positivity of delay differential and difference equations

In [Győri, Horváth 2017a] we studied Halanay type differential inequalities together with delayed differential inequalities of similar type. Following the example of the generalized characteristic function of non-autonomous delayed differential equations, we introduce the concept of generalized characteristic inequality, which is one of the fundamental tools of the studies. The other essential tool is to explore the close relationship between inequalities of two different forms. We have given constraints to the non-negative solutions of the inequalities studied and examined the conditions under which the non-negative solutions tend to zero. The applicability and sharpness of our results are illustrated by examples.

In [Győri, Horváth 2017b] we have given necessary and sufficient conditions for the non-negative solutions of certain delayed difference inequalities and Halanay type inequalities tending to zero. Delayed differential inequalities with constant arguments can be derived from the examined inequalities, to which the results of [Győri, Horváth 2017a] can be applied, and from which we can deduce that the non-negative solutions of the examined inequalities tend to zero. The main results of the paper supplement and significantly improve previous results.

In [Pituk 2017a], we proved a sufficient condition for the oscillation of the solutions of a first-order scalar linear delayed differential equation in the case where the coefficient function changes slowly at infinity. The result is interesting because the sufficient condition we have obtained is only slightly stronger than the classical necessary condition of Myshkis. Similar sharp oscillation criteria were previously known only for equations with constant and asymptotically constant coefficients.

According to our previous result for Poincaré difference equations, the rate of exponential growth of a positive solution is always a non-negative eigenvalue of the limit equation to

which a positive eigenvector belongs. Positivity is defined with respect to the partial order generated by a given order cone. The main theorem of [Pituk 2017b] is the continuous analogue of this result for linear ordinary differential equations with asymptotically constant coefficients. We have shown that the Lyapunov exponent of any positive solution is a real eigenvalue of the limit equation to which a positive eigenvector belongs. The proof is based on the result for difference equations and the variation of constants formula.

In [Awwad, Győri, Hartung, 2018] bounded input bounded output (BIBO) stability was investigated for a class of nonlinear difference equations with several time delays. We rewrite the closed loop system as an equivalent nonlinear Volterra difference equation (VDE) with delays. The BIBO stability results are based on our theorem which formulate sufficient conditions for the boundedness of the solutions of delayed VDEs. In cases when the nonlinearity of the equation has sublinear, linear or superlinear growth, explicit BIBO stability conditions are formulated.

[Győri, Horváth, 2018] was devoted to inhomogeneous Halanay-type inequalities together with inhomogeneous linear delay differential inequalities and equations. Based on the variation of constants formula and some results borrowed from our recent paper, sharp conditions for the boundedness and the existence of the limit of the nonnegative solutions were established. The sharpness of the results was illustrated by examples and by comparison of results in some earlier works.

In [Győri, Mohamady, Hartung, 2018a] a system of nonlinear delay differential equation is considered, and the persistence of the positive solutions is studied. The method uses a monotone iteration technique, so a key assumption is that some terms and some ratios of the parameters of the equations are monotonically increasing. In several particular cases, explicit formulas are given for the estimates of the upper and lower limit of the solutions. In a special case, the asymptotic equivalence of the solutions is investigated.

In [Győri, Mohamady, Hartung, 2018b] a class of scalar delay differential equation was considered with a mixed monotone nonlinear terms, both including delayed solutions. Sufficient conditions are formulated for the permanence of positive solutions. Our results gave lower and upper estimates of the limit inferior and the limit superior of the solutions via a special solution of an associated nonlinear system of algebraic equations. Several examples showed the applicability of the general results.

In [Győri, Nakata, Röst 2018] we studied bounded, unbounded and blow-up solutions of a delay logistic equation without assuming the dominance of the instantaneous feedback. It was shown that there can exist an exponential (thus unbounded) solution for the nonlinear problem, and in this case the positive equilibrium is always unstable. We obtained a necessary and sufficient condition for the existence of blow-up solutions, and characterized a large class of such solutions. There is a parameter set such that the non-trivial equilibrium is locally stable but not globally stable due to the co-existence of blow-up solutions.

In [Pituk 2018] a nonlinear perturbation of a linear autonomous retarded functional differential equation was considered. According to a Perron type theorem, with the possible exception of small solutions the Lyapunov exponents of the solutions of the perturbed equation coincide with the real parts of the characteristic roots of the linear part. In this paper, we studied those solutions which are positive in the sense that they lie in a given order cone in the phase space. The main result shows that if the Lyapunov exponent of a positive

solution of the perturbed equation is finite, then it is a characteristic root of the unperturbed equation with a positive eigenfunction. As a corollary, a necessary and sufficient condition for the existence of a positive solution of a linear autonomous delay differential equation was obtained.

It is known that if the coefficient matrix of a linear autonomous difference equation is nonnegative and primitive, then the solutions starting from nonnegative nonzero initial data are strongly ergodic. The strong ergodic property of the nonnegative solutions was earlier extended to equations with asymptotically constant coefficients. In [Pituk, Pötzsche, 2018], we presented a generalization of the previous results by showing that the nonnegative solutions satisfy a similar ergodic property also in some cases when the coefficient matrices are not asymptotically constant.

In [Győri, Horváth 2020b] we discussed the asymptotic properties of solutions of special inhomogeneous linear delay functional differential systems generated by nonnegative Volterra type operators. In the recent paper [Győri, Horváth 2020a] a variation of constants formula has been given for such systems, and this is particularly suited to handling the studied problems. First, we investigated the boundedness of the solutions. Then some estimates were given for the upper and lower limits of the solutions. By using this, we studied the existence of the limit of the solutions, and obtain a formula for the limit. The applicability of our results was illustrated by studying the dependence of limits of the solutions on the choice of inhomogeneities, synchronisation, and Lotka-Volterra type delay functional differential systems.

In [Pituk 2020] it was shown that if we exclude the existence of nontrivial small solutions, then a linear autonomous functional differential equation has a nontrivial nonnegative solution if and only if it has a nonnegative eigenfunction. The result was motivated by the oscillation theorem of functional differential equations. The method of proof followed similar lines as the proof of a Perron type theorem for positive solutions of a perturbed system of nonautonomous linear functional differential equations in out recent paper [Pituk 2018].

2. Gronwall-Bellman-Bihari and Jensen type inequalities

In [Horváth, 2018a] a well known operator Jensen's inequality was generalized to normal operators. The main techniques employed here were the spectral theory for bounded normal operators on a Hilbert space and different Jensen type inequalities. We emphasised the application of a vector Jensen's inequality. By applying our results, some classical inequalities obtained for self-adjoint operators could also be extended.

f-divergences play important role in probability theory, especially in information theory and in mathematical statistics. Remarkable divergences can be found among them. Inequalities for f-divergences are very useful and applicable in information theory. In [Horváth, Pecaric, Pecaric 2018] we gave a precise equality condition and a refinement for one of the basic inequalities of f-divergences. The results were illustrated by some applications.

In [Mehmood, Butt, Horváth; J. Pecaric, 2018] we generalized cyclic refinements of Jensen's inequality from a convex function to a higher-order convex function by means of Lagrange--Green's function and Fink's identity. We formulated the monotonicity of the linear

functionals obtained from these identities utilizing the theory of inequalities for n-convex functions at a point. New Grüss-and Ostrowski-type bounds were found for identities associated with the obtained inequalities. Finally, we investigated the properties of linear functionals regarding exponential convexity and mean value theorems.

The Jensen's inequality plays a crucial role to obtain inequalities for divergences between probability distributions. In [Horváth, Pecaric, Pecaric 2019], we introduced a new functional, based on the f-divergence functional, and then we obtained some estimates for the new functional, the f-divergence and the Rényi divergence by applying a cyclic refinement of the Jensen's inequality. Some inequalities for Rényi and Shannon entropies were obtained too. The Zipf-Mandelbrot law was used to illustrate the results.

In [Horváth 2019] Grüss type inequalities were studied for real and complex valued functions in probability spaces. Some earlier Grüss type inequalities were extended and refined. Our approach led to new integral inequalities which are interesting in their own right. As an application, we gave a Grüss type inequality for normal operators in a Hilbert space. Similar results were obtained only for self-adjoint operators in earlier papers.

The most important inequality for convex functions is the Jensen's inequality. Other inequalities such as inequalities for means, the Hölder's and Minkowski's inequalities etc. can be obtained as particular cases of it, and it has many applications in different branches of mathematics. There are countless papers dealing with generalizations, refinements, and converse results of Jensen's inequality. To give refinements of Jensen's inequality is an extensively investigated theme with numerous methods, results and applications. Our monograph [Butt, Horváth, Pecaric, Pecaric, 2020] is mainly devoted to cyclic refinements (cyclic permutations are used to define the refining terms), and their applications in information theory. It contains the most recent research results of this promising topic.

In [Horváth 2020] some new refinements of the discrete Jensen's inequality were obtained in real vector spaces. The idea comes from some former refinements determined by cyclic permutations. We essentially generalized and extended these results by using permutations of finite sets and bijections of the set of positive numbers. We get refinements of the discrete Jensen's inequality for infinite convex combinations in Banach spaces. Similar results are rare. Finally, some applications were given on different topics.

The main purpose of [Horváth 2021a] was to present essential extensions of results in former papers of the author, and apply them to some special situations. Of particular interest is the refinement of the integral Jensen inequality for vector valued integrable functions. The applications related to four topics, namely f-divergences in information theory (an interesting refinement of the weighted geometric mean - arithmetic mean inequality was obtained as a consequence), norm inequalities, quasi-arithmetic means, Hölder's and Minkowski's inequalities.

In [Horváth 2021b] we studied inequalities corresponding to Jensen-Mercer's inequality. Some new extensions of Niezgoda's inequality and the integral version of Jensen-Mercer's inequality were given. The obtained inequalities do not only generalize the former ones, but our proofs are natural and simple. They clearly show the structure of such inequalities: they consist of two parts, a discrete or integral Jensen's inequality and then a majorization type inequality. Another purpose of the paper was to provide a deeper understanding of the methods used to refine Jensen-Mercer's and the corresponding inequalities. Moreover, some

new refinements of these inequalities were obtained. Finally, some applications related to Fejér's and Hermite-Hadamard inequalities were given.

There are a lot of papers dealing with applications of the so called cyclic refinement of the discrete Jensen's inequality. A significant generalization of the cyclic refinement, based on combinatorial considerations, has recently been discovered by the author. In [Horváth 2021c] we gave the integral versions of these results. On the one hand, a new method to refine the integral Jensen's inequality was developed. On the other hand, the result contained some recent refinements of the integral Jensen's inequality as elementary cases. Finally, some applications to the Fejér inequality (especially the Hermite-Hadamard inequality), quasi-arithmetic means, and f-divergences were presented.

3. State-dependent functional differential equations

In [Hartung 2021] a class of nonlinear neutral differential equation with state-dependent delays was considered, and well-posedness of the initial value problem and differentiability of the solutions with respect to parameters were investigated. To prove our main results we assumed the strict monotonicity of the time lag functions of the state-dependent delays, and that the state-dependent delay in the neutral term is bounded below by a positive constants. The differentiability was proved using a certain Sobolev norm for the space of the initial functions. We note that similar smoothness results for state-dependent neutral differential equations were proved only for more special cases.

4. Other results related to differential and difference equations

In [Lipták, Hangos, Pituk, Szederkényi, 2018], we introduced a class of delayed kinetic systems derived from mass action type reaction network models. We defined the time delayed positive stoichiometric compatibility classes and the notion of complex balanced time delayed kinetic systems. We proved the uniqueness of equilibrium solutions within the time delayed positive stoichiometric compatibility classes for such models. In our main result we proved the semistability of the equilibrium solutions for time delayed complex balanced systems using an appropriate Lyapunov-Krasovskii functional and LaSalle's invariance principle. As a consequence, we obtained that every positive complex balanced equilibrium solution is locally asymptotically stable relative to its positive stoichiometric compatibility class. Our paper was awarded by Advances in Engineering as a 'Key Scientific Article'.

A linear autonomous differential equation with small delay was considered in [Fehér, Márton, Pituk 2019]. It was shown that under a smallness condition the delay differential equation is asymptotically equivalent to a linear ordinary differential equation with constant coefficients. The coefficient matrix of the ordinary differential equation is a solution of an associated matrix equation and it can be written as a limit of a sequence of matrices obtained by successive approximations. The eigenvalues of the approximating matrices converge exponentially to the dominant characteristic roots of the delay differential equation and an explicit estimate for the approximation error was given.

The variation of constants formula is one of the principal tools of the theory of differential equations. There are many papers dealing with different versions of the variation of constants

formula and its applications. Our main purpose in [Győri, Horváth 2020a] was to give a variation of constants formula for inhomogeneous linear functional differential systems determined by general Volterra type operators with delay. Our treatment of the delay in the considered systems was completely different from the usual methods. We dealt with the representation of the studied Volterra type operators. Some existence and uniqueness theorems were obtained for the studied linear functional differential and integral systems. Finally, some applications were given.

In [Federson, Győri, Mesquita, Táboas, 2020] an autonomous impulsive system was proposed as a model of drugs absorption by living organisms consisting of a linear differential delay equation and an impulsive self-support condition. We got a representation of the general solution in terms of the fundamental solution of the differential delay equation. The impulsive self-support generated periodic auto-oscillations given by fixed points of a first return map.

In [Dragičević, Pituk 2021] we established sufficient conditions under which a large class of semilinear nonautonomous difference equations with infinite delay is Hyers-Ulam stable. These conditions require that the nonautonomous linear part admits an exponential dichotomy and the nonlinear perturbations are uniformly Lipschitz continuous with a sufficiently small Lipschitz constant. In the more general case when the linear part admits a shifted exponential dichotomy, we gave sufficient conditions for the existence of a certain weighted form of the shadowing property.

The problem of finding the oscillation bounds for first order linear delay differential equations has been in the focus of the oscillation theory for more than 30 years. Although numerous estimates for the oscillation bounds were available in the literature, their explicit values were not known. In [Pituk, Stavroulakis Stavroulakis 2021] we have given the oscillation bounds explicitly in terms of the real branches of the Lambert W function.

Other scientific activities

Ferenc Hartung and Mihály Pituk have organized the international conference „Veszprém Conference on Differential and Difference Equations and Applications (July 2-5, 2018, Veszprém, Hungary)".

Other supports

The research group has received the OTKA grant KH130513 (2018-2021). The topic of this project had no overlap with that of K120186, and it resulted in 10 scientific publications (not listed in this report).

I consent to the publication of the final report of the grant K120186 by the OTKA Committee in an accessible archive known to the scientific community.

Veszprém, December 23, 2021.

Dr. Ferenc Hartung, principal investigator

List of publications

1. Awwad E, Győri I, Hartung F: ***BIBO stability of discrete control systems with several time delays***, Miskolc Mathematical Notes 19:1, 95-109, 2018, Rank: Q3, IF=0.585, independent citations=1
2. Butt S.I., Horváth L., Pecaric D, Pecaric J: ***Cyclic Improvements of Jensen's Inequalities - Cyclic Inequalities in Information Theory***, MONOGRAPHS IN INEQUALITIES 18, Element, Zagreb, ISBN 978-953-197-686-2, 2020
3. Dragičević D., Pituk M.: ***Shadowing for nonautonomous difference equations with infinite delay***, Applied Mathematics Letters 120, article no. 107284, 2021, Rank: Q1, IF=4.055, independent citations=1
4. Fehér Á., Márton L., Pituk M.: ***Approximation of a Linear Autonomous Differential Equation with Small Delay***, Symmetry, 11, (10 p.) 1299, doi:10.3390/sym11101299, 2019, Rank: Q2, IF=2.645, independent citations=2
5. Federson M., Győri I., Mesquita, J. G., Táboas, P.: ***A Delay Differential Equation with an Impulsive Self-Support Condition***, J Dyn Diff Equat 32, 605–614, 2020, Rank: Q1, IF=1.473, independent citations=1
6. Győri I., Hartung F, Mohamady N. A.: ***Boundedness of positive solutions of a system of nonlinear delay differential equations***, Discrete and Continuous Dynamical Systems - Series B, 23:2, 809-836, 2018, Rank: Q2, IF=1.008, independent citations=2
7. Győri I., Hartung F, Mohamady N. A.: ***Permanence in a class of delay differential equations with mixed monotonicity***, Electronic Journal of Qualitative Theory of Differential Equations 2018:53, 1-28, 2018, Rank: Q2, IF=1.065, independent citations=17
8. Győri I., Horváth L.: ***Sharp estimation for the solutions of delay differential and Halanay type inequalities***, Discrete Contin. Dyn. Syst., 37:6, 3211-3242, 2017, Rank: D1, IF=0.967, independent citations=6
9. Győri I., Horváth L.: ***Connection Between Continuous and Discrete Delay and Halanay type Inequalities***, in "Advances in Difference Equations and Discrete Dynamical Systems" ICDEA, Osaka, Japan, July 2016, Springer Proceedings in Mathematics & Statististics 212, pp. 91-112, 2017
10. Győri I., Horváth L.: ***Sharp estimation for the solutions of inhomogeneous delay differential and Halanay type inequalities***, Electron. J. Qual. Theory Differ. Equ., No. 54, 1-18, 2018, Rank: Q2, IF=1.065, independent citations=3
11. Győri I., Horváth L.: ***On the fundamental solution and its application in a large class of differential systems determined by Volterra type operators with delay***, Discrete Contin. Dyn. Syst. A, 40, No. 3., 1665-1702, 2020, Rank: D1, IF=1.392
12. Győri I., Horváth L.: ***Explicit estimates and limit formulae for the solutions of linear delay functional differential systems with nonnegative Volterra type operators***, Appl. Math. Comput., 385, 125451, 2020, Rank: Q1, IF=3.472, independent citations=1
13. Győri I, Nakata Y., Röst G.: ***Unbounded and blow-up solutions for a delay logistic equation with positive feedback***, Communications on Pure & Applied Analysis, 17 (6), 2845-2854, 2018, Rank: Q1, IF=0.925
14. Hartung F.: ***Differentiability of Solutions with respect to Parameters in a Class of Neutral Differential Equations with State-Dependent Delays***, Electronic Journal of Qualitative Theory of Differential Equations, No. 56, 1–41, 2021, Rank: Q2, IF=1.874
15. Horváth L: ***Generalizations of Jensen's operator inequality for convex functions to normal operators***, Ann. Funct. Anal., 9, No. 4., 566-573., 2018, Rank: Q2, IF=0.577, independent citations=1
16. Horváth L.: ***Grüss type and related integral inequalities in probability spaces***, Aequat. Math., 93, No. 4., 743-756, 2019, Rank: Q2, IF= 0.851
17. Horváth L.: ***New refinements of the discrete Jensen's inequality generated by finite or infinite permutations***, Aequat. Math., 94, 1109-1121, 2020, Rank: Q2, IF= 0.832, independent citations=1

18. Horváth L.: ***Extensions of recent combinatorial refinements of discrete and integral Jensen inequalities***, Aequationes Mathematicae, https://doi.org/10.1007/s00010-021-00821-x, 2021, Rank: Q2, IF= 0.832
19. Horváth L.: ***Some notes on Jensen-Mercer's type inequalities; extensions and refinements with applications***, Mathematical Inequalities & Applications, v24:4, 1093-1111, 2021, Q2, IF= 1,250
20. Horváth L.: ***Refinements of the integral Jensen's inequality generated by finite or infinite permutations***, Journal of Inequalities and Applications, 2021:1, paper 12, 2021, Rank: Q1, IF= 2.491
21. Horváth L., Pečarić Đ, Pečarić J.: ***A Refinement and an Exact Equality Condition for the Basic Inequality of f -divergences***, Filomat 32:12, 4263–4273, 2018, Rank: Q2, IF= 0.789
22. Horváth L., Pečarić Đ, Pečarić J.: ***Estimations of f- and Rényi divergences by using a cyclic refinement of the Jensen's inequality***, Bull. Malays. Math. Sci. Soc., 42, No. 3., 933-946, 2019, Rank: Q3, IF= 0.856, independent citations=9
23. Lipták Gy, Hangos K M, Pituk M, Szederkényi G: ***Semistability of complex balanced kinetic systems with arbitrary time delays***, Systems & Contol Letters 114, 38-43, 2018, Rank: D1, IF= 2.624, independent citations=11
24. Mehmood N, Butt S I, Horváth L, Pečarić J: ***Generalization of cyclic refinements of Jensen inequality by Fink's identity***, J. Inequal. Appl., 2018:51, 21 pp, 2018, Rank: Q3, IF= 1.136, independent citations=3
25. Pituk M.: ***A Corollary of a Theorem on Positive Solutions of Poincaré Difference Equations***, in "Advances in Difference Equations and Discrete Dynamical Systems" ICDEA, Osaka, Japan, July 2016, Springer Proceedings in Mathematics & Statististics 212, pp. 199-205, 2017
26. Pituk M.: ***Oscillation of a linear delay differential equation with slowly varying coefficient***, Applied Mathematics Letters, 73, pp. 29-36, 2017, Rank: Q1, IF=2.462, independent citations=11
27. Pituk M: ***A Perron type theorem for positive solutions of functional differential equations***, Electronic Journal of Qualitative Theory of Differential Equations, No. 57, 1-11, 2018, Rank: Q2, IF= 1.065, independent citations=2
28. Pituk M.: ***Existence of Nonnegative Solutions of Linear Autonomous Functional Differential Equations***, Mathematics, 8(7), 1098; https://doi.org/10.3390/math8071098, 2020, Rank: Q2, IF= 2.258, independent citations=1
29. Pituk M, Pötzsche C: ***Ergodicity beyond asymptotically autonomous linear difference equations***, Applied Mathematics Letters 86, 149–156, 2018, Rank: Q1, IF= 3.487
30. Pituk M., Stavroulakis I.P. and Stavroulakis J.I.: ***Explicit values of the oscillation bounds for linear delay differential equations with monotone argument***, Communications in Contemporary Mathematics, article no. 2150087, 2021, Rank: Q1, IF= 1.665