Closing report: Rings, semigroups, categories

We did research in various topics of ring theory, semigroup theory, and category theory, such as invariant theory, polynomial identities, matrices over noncommutative rings, Morita theory, quantum groups, tropical algebra, cohomology theory of real flag manifolds, homotopical algebra. 35 of our papers dedicated to the present project have appeared or are accepted for publication, most of them in international journals, and a couple in refereed book chapters. An overview of the majority of these papers is given below (for a complete list of publications see the summary form).

Our research group consisted of 4 senior participants (including the principal investigator) and 6 other researchers (two of them joined us in the third year). Three of the non-senior participants defended their PhD during the project (one of them under the supervision of the principal investigator). In the last 3 months of the project we included three university students to take part in our research work in the form of a summer internship.

Invariant Theory. The Noether number of a finite group G is the minimal positive integer d such that all rings of invariants of G are generated by elements of degree at most d. There are very few groups for which the exact value of the Noether number is known. In [7] the Noether numbers of all groups of order less than 32 were determined. It turned out that for these groups the Noether number is attained on a multiplicity free representation, it is strictly monotone on subgroups and factor groups, and it does not depend on the characteristic (provided that it does not divide the group order). For an abelian group G, the Noether number coincides with the Davenport constant, the maximal length of an irreducible zero-sum sequence over G. A non-abelian group has a small and a large Davenport constant, and Geroldinger and Grynkiewicz asked if the Noether number is always between the two. We developed and implemented an algorithm to compute these Davenport constants. It turned out that for groups of order less than 32, the Noether number is always greater than or equal to the small Davenport constant, and with the only exception of the Heisenberg group of order 27, it is less than or equal to the large Davenport constant.

The best known method to give a lower bound for the Noether number of a given finite group is to use the fact that it is greater than or equal to the Noether number of any of the subgroups or factor groups. The observation from [7] that the Noether number is strictly monotone for subgroups of groups of order less than 32 was extended in [8] to a general theorem, asserting that the above mentioned inequalities are always strict for proper subgroups or factor groups. This is established by studying the algebra of coinvariants of a representation induced from a representation of a subgroup.

As mentioned above, results on zero-sum sequences over a finitely generated abelian group can be translated to statements on generators of rings of invariants of the dual group. The direction of the transfer of information between zero-sum theory and invariant theory was reversed in [9]. First it was shown how a presentation by generators and relations of the ring of invariants of an abelian group acting linearly on a finite-dimensional vector space can be obtained from a presentation of the ring of invariants for the corresponding multiplicity free representation. This combined with a known degree bound for syzygies of rings of invariants yields bounds on the presentation of a block monoid associated to a finite sequence of elements in an abelian group.

A separating system of polynomial invariants of a finite group (with a given linear representation) is a set of polynomial invariants that are sufficient to separate the orbits. A generating system is always a separating system, and the study of this notion became popular in the past two decades. In particular, the separating Noether number was introduced in [11] in analogy with the Noether number. In view of the fact that for an abelian group, the Noether number coincides with the Davenport constant, it was natural to look for a characterization of the separating Noether number of an abelian group in terms of additive combinatorics. This was achieved in [10], and as an application of the result, it was proved that the separating Noether number of a finite abelian group is almost always strictly smaller than its Noether number. The latter result shows that the relaxation of the condition "generating" to "separating" is reflected even in degree bounds.

The main result of [6] is that for any non-cyclic p-group G and any base field of characteristic greater than p the Noether number of G is at least $|G|/p$. The main intermediary result from which this conclusion follows is the proof of the fact that for the extraspecial group of order p^3 (the Heisenberg group) the Noether number is less than p^2, when $p > 3$ and it equals 9 for $p = 3$.

The algebra of invariants of the special orthogonal group acting on tuples of vectors is well understood both from a combinatorial or representation theoretic point of view (at least in characteristic zero). This information was used in [12] to compute the cocharacter sequence of the weak polynomial identities of 3×3 skew-symmetric matrices. The cocharacter sequence is a central quantitative invariant in the theory of polynomial identities, and as far as the polynomial identities of representations of a Lie algebra are concerned, similar exact results were known before only for the Lie algebra of 2×2 traceless matrices with its defining 2-dimensional representation.

Vector invariants of the orthogonal group have a satisfactory description also in odd positive characteristic by [13]. However, a minimal generating system is not known in characteristic 2, when some exotic new invariants were found in [14]. Now we gave a characteristic free description of semi-invariants of tuples of 2×2 matrices in [16]. As an application a minimal homogeneous system of generators of the algebra of the vector invariants of the special orthogonal group is obtained in characteristic 2 in the 4-dimensional case (the general case is still open). It turns out that the classical invariants together with the new ones found in [14] do generate in this case.

Polynomial Identities. A theorem of Kaplansky [17] asserts that if a finitely generated associative algebra satisfies the polynomial identity $x^n = 0$, then it is nilpotent. From known results in [18] and [19] we deduced in [16] that an m-generated nil algebra of bounded nil-index n is nilpotent with nilpotency index at most $(m+1)n^4$. This is a drastic improvement of the bounds known before for the nilpotency index. The lower bound from [20] was also extended to all characteristic in [16].

A possibility to develop noncommutative invariant theory is to replace the commutative polynomial algebra by a relatively free algebra (a factor of the tensor algebra modulo an ideal stable under all algebra endomorphisms). This topic was popular for a few years in the 1980ies, but produced mainly negative results in the sense that they indicated that algebras of invariants are typically not finitely generated. Whenever a commutative graded subalgebra of the polynomial algebra is finitely generated, it has a rational Hilbert series by the so-called Hilbert-Serre theorem. We showed in [21] that despite the fact that the subalgebras of invariants in noncommutative relatively free algebras are rarely finitely generated, they always have a rational Hilbert series for representations of reductive groups or maximal unipotent subgroups of reductive groups (the most relevant groups considered in invariant theory).

The class of relatively free algebras where reductive groups have finitely generated subalgebras of invariants is the class of Lie nilpotent relatively free algebras. In [22] a constructive approach was given to compute the generators in this case. More precisely, it is explained how one can reduce the problem to a commutative invariant theory situation, where the methods of constructive commutative invariant theory work. A key step in the process concerns a more general situation: finding generators of the subalgebra of invariants a group of automorphisms of a Lie algebra acting on the universal enveloping algebra.

Let R be a Lie nilpotent algebra of index $k \geq 1$ over a field K of characteristic zero. If G is an n-element subgroup $G \subseteq \mathrm{Aut}_K(R)$ of K-automorphisms. It is proved in [53] that R is right integral over $\mathrm{Fix}(G)$ of degree n^k. In the presence of a primitive n-th root of unity $e \in K$, for a K-automorphism $\delta \in \mathrm{Aut}_K(R)$ with $\delta^n = \mathrm{id}_R$, we prove that the skew polynomial algebra $R[w, \delta]$ is right integral of degree n^k over $\mathrm{Fix}(\delta)[w^n]$.

In 1969, R. G. Swan gave a graph-theoretic proof of the Amitsur–Levitzki theorem which states that the standard identity of degree $2n$ holds for the ring of $n \times n$ matrices over a commutative ring.

In [40] we extended this theorem to the case where the commutative ring is replaced by a finite-dimensional Grassmann algebra. The arguments are purely combinatorial, based on computing sums of signs corresponding to Eulerian trails in directed graphs. Before the current result, the gap between available lower and upper bounds was relatively wide: quadratic from above, linear from below. The theorem fully settles the 2-generated and 3-generated cases, and provides an upper bound for all finite-dimensional Grassman algebras, that is conjectured to be optimal in all cases. Computer simulation supports the conjecture.

Quantum Groups. We have examined two related structures: the quantized coordinate ring of the special linear group – which is a non-commutative algebra –, and the corresponding semi-classical limit Poisson structure on the commutative coordinate ring of the special linear group. The subalgebra of adjoint action invariants in the coordinate ring of the special linear group is generated by the coefficients of the characteristic polynomial. It was proved earlier in [41] that in the quantized case, the ring of invariants under the adjoint coaction is generated by quantum analogues of these coefficients, and coincides with the subalgebra of the cocommutative elements. Moreover, this subalgebra is commutative. Now we proved that this is a *maximal* commutative subalgebra in the quantized case (see [42]), whereas In the classical commutative coordinate ring, the subalgebra of adjoint action invariants is a a maximal Poisson-commutative subalgebra (see [43]). In fact in both cases these subalgebras are centralizers (in the appropriate sense) of a single element: the trace. Maximal Poisson-commutative subalgebras in Poisson-algebras are of interest because of their relations to integrable systems, studied by physicists.

Matrix Theory. In [54] we proved that if F is any field and R is any F-subalgebra of the algebra $\mathbb{M}_n(F)$ of $n \times n$ matrices over F with Lie nilpotence index m, then

$$\dim_F R \leqslant M(m+1, n)$$

where $M(m+1,n)$ is the maximum of $\frac{1}{2}\left(n^2 - \sum_{i=1}^{m+1} k_i^2\right) + 1$ subject to the constraint $\sum_{i=1}^{m+1} k_i = n$ and $k_1, k_2, \ldots, k_{m+1}$ nonnegative integers. The case $m = 1$ reduces to a classical theorem of Schur (1905), later generalized by Jacobson (1944) to all fields, which asserts that if F is an algebraically closed field of characteristic zero, and R is any commutative F-subalgebra of $\mathbb{M}_n(F)$, then $\dim_F R \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor + 1$. Examples constructed from block upper triangular matrices show that the upper bound of $M(m+1,n)$ cannot be lowered for any choice of m and n. An explicit formula for $M(m+1,n)$ is also derived.

For an $n \times n$ matrix A over a Lie nilpotent ring R of index k, with $k \geq 2$, we prove in [55] that an invariant "power" Cayley-Hamilton identity

$$\left(I_n \lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^{2^{k-2}} = 0$$

of degree $n^2 2^{k-2}$ holds. The right coefficients $\lambda_i^{(2)} \in R$, $0 \leq i \leq n^2$ are not uniquely determined by A, and the cosets $\lambda_i^{(2)} + D$, with D the double commutator ideal $R[[R,R],R]R$ of R, appear in the so-called second right characteristic polynomial $p_{\overline{A},2}(x)$ of the natural image $\overline{A}$ of A in the $n \times n$ matrix ring $\mathrm{M}_n(R/D)$ over the factor ring R/D:

$$p_{\overline{A},2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2}.$$

First we exhibit in [56] two $n \times n$ matrices generating the full $n \times n$ matrix algebra as a Lie algebra (it is much stronger than associative generation). There is a chance to use these two matrices to provide an explicit description of the Lie automorphisms of the full matrix algebra. The notion of the Lie centralizer of a subset in an associative algebra is introduced. Some results are obtained about the so called Lie centralizers in general associative algebras. If K is any field of characteristic different from 2 and 3, then every Lie automorphism ψ of $\mathrm{M}_n(K)$ can be presented as a sum

$$\psi = \sigma + \tau,$$

where σ is either an automorphism of $\mathrm{M}_n(K)$ (as a K-algebra) or the negative of an anti-automorphism of $\mathrm{M}_n(K)$, and τ is an additive mapping from $\mathrm{M}_n(K)$ to K which maps commutators into zero. In the light of this significant result due to Martindale, we present a unifying approach to constructively describe automorphisms and anti-automorphisms of $\mathrm{M}_n(K)$.

General Ring Theory. It is a nice, simple result of Kaplansky that a commutative ring is von Neumann regular if and only if all its simple modules are injective. However, the corresponding statement is no longer true for noncommutative associative rings. This leads to the study of V-rings which are not von Neumann regular but still have simple injective modules. One old guiding problem in the study of V-rings is a question of C. Faith. We answered in [1] positively this conjecture by constructing V-domains with arbitrarily prescribed finite numbers of isomorphism classes of simple modules. The solution is a tricky application of Gabriel's flat localizations.

It is well-known that results from field theory are useful in linear algebra for a structural description of linear transformations. In [4] we show that the converse process is also fruitful with applications in the foundations of field theory. Namely, the basic Cayley-Hamilton theorem that the companion matrix is a solution of its defining equation, can be used to construct simple algebraic extensions. This approach makes the discussion of uniqueness of simple algebraic extensions transparent. In this way one can "identify" easily algebraic closures of a field inside endomorphism rings of possibly infinite dimensional vector spaces, making a common umbrella for both linear algebra and commutative-associative ring theory.

In [2] we investigate commutative rings whose principal ideals have unique generators. This treatment led to several interesting quasi-varieties of commutative rings where Mersenne primes appear naturally. This approach provides therefore useful, elementary examples for quasi-varieties missing in the study of equational classes in universal algebra.

Every idempotent e in an algebra A induces a Peirce decomposition $A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e)$ which is equivalent to say that A is a formal generalized matrix ring $A = \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}$ which leads naturally to a notion of n-Peirce rings. In [3] we describe the structure of n-Peirce rings and show that the Peirce dimension n is an important invariant in the study of rings. We compute the automorphism group of n-Peirce rings in terms of their ingredients.

In [5] we describe certain irreducible representations of Leavitt path algebras which are recently a subject of intensive study, by using appropriate bases as well as their defining relations. By this way it is easy to determine when they are finitely presented as well as to compute associated invariants like their endomophism rings and annihilator primitive ideals.

In [52] we provide an elementary proof of the Skolem-Noether theorem, which is entirely constructive and gives the conjugating matrix explicitly, using the images of only two generating matrices, a nonzero vector in a certain kernel, and matrix multiplication.

Semigroup Theory. The notion of firm semigroups has been introduced and the theory of Morita equivalence for such semigroups has been developed in [45]. It was shown that the categories of acts used by earlier authors in various classes of semigroups for developing Morita equivalence are all equivalent to each other for firm semigroups (whose class contains all the other classes of semigroups in which Morita equivalence has been considered so far), hence they yield the same Morita equivalence. This connects several earlier results whose relation to each other has not been clear so far. It has also been proved that Morita equivalence between any two firm semigroups can be obtained from a Morita context connecting the two semigroups - till now this was known only for semigroups with local units, which is a much narrower class. These results show that the class of firm semigroups is the natural framework for investigating Morita equivalence of semigroups.

Properties of the lattice of unitary ideals of a semigroup have been studied in [46]. In particular, it has been shown that this lattice forms a quantale. It has been proved that if two semigroups are connected by an acceptable Morita context then there is an isomorphism between the quantales of unitary ideals of these semigroups. Moreover, factorisable ideals corresponding to each other under this isomorphism are strongly Morita equivalent.

A general notion of quotient ring based on inverses along an element has been introduced in [47]. This notion encompasses quotient rings constructed using various generalized inverses. On the other hand, such quotient rings can be viewed as Fountain-Gould quotient rings with respect to appropriate subsets, hence the variant of the Fountain-Gould construct introduced in an earlier paper by Ánh and Márki yields the key notion for quotient rings of various kinds. The connection between partial order relations on a ring and on its ring of quotients has also been investigated.

Radical Theory. Combinatorial exactness structures of several levels have been introduced earlier for an abstract presentation of Kurosh-Amitsur radical theory. Each of these structures contains a distinguished point, which corresponds to the zero element. Now the one- and the two-dimensional exactness structures have been extended in [44] to the case with no distinguished point. This new two-dimensional combinatorial exactness structure allows to define a radical-semisimple triple in such a way that if (R,r,S) is a radical-semisimple triple, then (R,S) is a radical-semisimple pair with respect to its underlying one-dimensional exactness structure. Using this, it has been shown among others that topological closure is a special case of the radical function in our sense.

Tropical Algebra. The *tropical semifield* $\mathbb{T}$ is the set $\{\mathbb{R} \cup \{-\infty\}\}$ with two operations: maximum playing the role of addition and addition acting as multiplication. In tropical geometry certain polyhedral complexes in $\mathbb{T}^n$, called *tropical varieties* are assigned to algebraic varieties via valuation maps. These provide a wealth of combinatorial tools to approach classical problems. Some applications include computing Gromov-Witten invariants due to Mikhalkin in [34], tropical proof of the Brill-Noether Theorem [24], Brill-Noether theory for curves of a fixed gonality [29], developing a strategy to attack the Riemann hypothesis [25], the Gross-Siebert program in mirror symmetry [27], and studying toric degenerations [32]. Tropical varieties can be described as a set where certain polynomial equations of the tropical polynomial semiring $\mathbb{T}[x_1, \ldots, x_n]$ hold. Hence there has been a natural interest in studying the algebraic properties of $\mathbb{T}[x_1, \ldots, x_n]$ (see for example [23], [26]. [28], [33]) and in particular in finding an appropriate notion of algebraic dimension (see for example [35]). Our work consisted of two parts. First in [30] we established a notion of primeness for congruences of arbitrary semirings, gave a full description of these primes for polynomial semirings over the tropical and the Boolean semifields and finally applied these results to establish a Nullstellensatz which is a strong generalization of the results in [23]. Secondly in [31] we introduced a notion of Krull-dimension and showed that it carries some natural algebraic properties as well as that it can be used to recover the usual dimension of tropical varieties.

Enumerative algebraic geometry over the reals. There is a class of linear enumerative geometry problems called *Schubert calculus*: enumerative problems that can be formulated in terms of linear subspaces intersecting each other in given dimensions. In the complex case Schubert problems are completely described by taking the product of some Schubert cycles $[\sigma_i]$ in the cohomology ring of a complex flag manifold; this is classical and several combinatorial characterizations are known [38]. In contrast, the solution to a real Schubert problem is not a single number, as it depends on the generic configuration. The number of possible solutions is not well-understood and is an active topic of research [50], [39], [36], [48], [51].

In the real case, the cohomological computation also makes sense and gives a lower bound to the number of possible solutions. In our work, we determined the cohomology rings of flag manifolds: first, we determined the incidence coefficients in the Vassiliev complex which can be used to determine the additive structure [49], and we also determined the multiplicative structure using equivariant cohomology in [37]. In particular, we obtained the following theorem: There is a degree-halving isomorphism of rings, which maps $[\sigma^{\mathbb{R}}_{D\lambda}]$ to $[\sigma^{\mathbb{C}}_{\lambda}]$:

$$H^*(\mathrm{Gr}_{2k}(\mathbb{R}^{2n}); \mathbb{Q}) \cong H^*(\mathrm{Gr}_k(\mathbb{C}^n); \mathbb{Q})$$

A similar degree-halving ring isomorphism $[\sigma^{\mathbb{R}}_{DI}] \mapsto [\sigma^{\mathbb{C}}_{I}]$ exists for even flag manifolds, i.e. when all subspaces have even dimension. In terms of enumerative problems, we obtain a lower bound to any even dimensional real Schubert problem. For instance, we obtain that the number of 8-planes in $\mathbb{R}^{16}$ intersecting four given 8-planes in 4 dimensions is at least 6 and at most 70, where the latter is the number of complex solutions.

Another related result in [49] concerning the integer coefficient cohomology of real even flag manifolds is that every torsion element has order exactly 2. This is shown using the incidence coefficients, which are related to the first Steenrod square of Schubert varieties. The first Steenrod square in turn was used to show degeneration of the Bockstein Spectral Sequence of even real flag manifolds, which proves the result.

Homotopical Algebra. Let X be a scheme and n a positive integer. We can view vector bundles of rank n as *forms* of the free $\mathcal{O}_X$-module of rank n. That is, an $\mathcal{O}_X$-module E is a vector bundle of rank n, if there exists a Zariski covering $U \to X$ and an isomorphism of $\mathcal{O}_U$-modules $E|U \cong \mathcal{O}_U^{\oplus n}$. The *Hilbert 90 Theorem* [58] shows that it is equivalent to require a trivializing fppf-covering. If we have a family of vector bundles, we can always describe its limit as a perfect complex. The following generalization for perfect complexes of the above result was shown in [57]: Let $X \xrightarrow{f} S$ be a proper morphism of quasi-compact and quasi-separated schemes, and E, F perfect complexes on X. Then the stack of families of morphisms $\mathcal{H}om_{X/S}(E, F)$ is algebraic.

In particular, one can use deformation theory in the automorphism group stack $\mathcal{A}ut_{X/S}(E)$. With one such deformation theory argument, a Hilbert 90-type result in this setting was also proved in [57]: Let X be a Noetherian scheme with infinite residue fields, and E a perfect complex on X. Suppose that F is an fppf-form of E, that is there exists an fppf covering $U \to X$ and a quasi-isomorphism $E|U \simeq F|U$. Then F is a Zariski form of E, that is there exists a Zariski covering $V \to X$ and a quasi-isomorphism $E|V \simeq F|V$.

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