ARRANGEMENTS AND APPROXIMATIONS OF CONVEX BODIES

FINAL REPORT OF PROJECT K119670 BY MÁRTON NASZÓDI

The project, which ran from December, 2016 until October, 2022, consisted of research by the PI, Márton Naszódi and by Zsolt Lángi, who joined essentially at the start, in April, 2017. It supported the publication of 38 papers that can be classified into the following areas: Quantitative Helly-type problems (discussed in Section 2), Approximation questions (see Section 3), Coverings (see Section 4), Arrangements of convex bodies (see Section 5), Volumetric inequalities (see Section 6) and Planar configurations (see Section 7).

A common theme in these works is the search for combinatorial and analytic ideas that help tackle questions that originate in discrete geometry or that have a flavor of, or analogy in discrete geometry, and vica versa, studying how geometric concepts yield results in algorithmic/computational and analytical questions.

We start with a detailed description of the results that are the most significant in our view in Section 1. In the sections following it, we present one or two results that we have achieved in the given topic. We finish our report with Section 8 detailing how the project contributed to the training of new, talented researchers of the field.

1. The most notable results

1.1. The problem of **efficient approximation of a convex body by a convex polytope**, a central question in convexity, is an instance of the general mathematical theme of discretization. In this classical field, the questions vary according to the measure of efficiency (few facets, or few vertices, etc.), the notion of distance with respect to which the approximation is to take place (Hausdorff distance, Banach–Mazur distance, volume of the difference, etc.), as well as the family of admissible convex polytopes (inscribed, circumscribed, etc.).

First, consider fine approximation in the so called *geometric distance* (a close relative of the *Banach–Mazur distance*), that is, we are looking for a convex polytope P contained in a fixed convex body K, with as few vertices as possible, whose slightly magnified copy $(1+\varepsilon)P$ contains K. Dmitry Ryabogin, Fedor Nazarov and the PI [21] showed the following.

Theorem 1 (D. Ryabogin, F. Nazarov, MN [21]). *For every convex body K in $\mathbb{R}^d$ with the center of mass at the origin and every $\varepsilon \in \left(0, \frac{1}{2}\right)$, there exists a convex polytope P with at most $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.*

This result improves the 2012 theorem of A. Barvinok [Bar14] by removing the assumption that K is centrally symmetric, and the extraneous $\left(\log \frac{1}{\varepsilon}\right)^d$ factor. The proof uses a mixture of geometric and probabilistic tools. By taking K to be the Euclidean ball, one sees quite easily that the order of magnitude for the number of vertices is best possible.

Our paper [21] has received 4 independent citations according to Google Scholar.

The following rough approximation result (that is, fewer vertices, worse bound on the distance), also in terms of the geometric distance, proved by the PI [27] relies on the application of a combinatorial tool.

Theorem 2 (MN [27])**.** *Roughly* $\frac{d}{(1-\vartheta)^d} \ln \frac{1}{(1-\vartheta)^d}$ *points* chosen uniformly and independently *from a convex body* K *in* $\mathbb{R}^d$*, whose center of mass is* o*, yield a polytope* P *for which* $\vartheta K \subseteq P \subseteq K$ *holds with large probability.*

Note that here, we are not free to choose how we construct the polytope P, it is the convex hull of a uniform sample of points from K of a certain size. This gives a joint generalization of results of Brazitikos, Chasapis and Hioni [BCH16] and of Giannopoulos and Milman [GM00]. The power of this result is the simplicity of its proof, and the method equally applies to studying half-space depth with respect to arbitrary measures. Our paper [27] is the basis of Chapter 9 in N. Mustafa's textbook *Sampling in combinatorial and geometric set systems* [Mus22].

1.2. F. John's foundational result on the characterization of the **largest volume ellipsoid contained in a convex body by an ellipsoid** [Joh14] has seen several applications and extensions since its publication in 1948. Recently, generalizations of classical results in convexity to the realm of **logarithmically concave functions**, that is, functions of the form $f = e^{-\psi}$, where $\psi : \mathbb{R}^d \longrightarrow (-\infty, \infty]$ is a convex function, have received attention. It is natural to ask if John's result can be extended. The first steps were made by D. Alonso-Gutiérrez, B. Merino, J. Jiménez and R. Villa in [Alo+18], where a functional analogue is proposed: Given a log-concave function f, we are looking for a function g which comes from a certain family $\mathcal{E}$ of log-concave functions (call them *functional ellipsoids*), is pointwise smaller than f and has the largest integral among functions satisfying these constraints. In [Alo+18], $\mathcal{E}$ is defined as the family of positive multiples of characteristic functions of ellipsoids in $\mathbb{R}^d$, and several properties of the solution of the above extremum problem are shown.

Grigory Ivanov and the PI [8] replaced the family $\mathcal{E}$ with a one-parameter sequence of familes $\mathcal{E}_s$ (with $s \in (0, \infty]$), and obtained a more complete picture: First, we can treat the normal distribution as a functional ellipsoid (the $s = \infty$ case), second we recover the results of [Alo+18] (the $s \to 0$ case), and most importantly, we now have a condition of optimality similar to John's theorem, which was missing from [Alo+18].

As an application of our John-type condition of optimality, we proved the following quantitative Helly-type result for log-concave functions, a functional analogue of the *Volume Helly Theorem*, see page 3.

Theorem 3 (G. Ivanov, MN [8])**.** *Let* $f_1, \ldots, f_n$ *be upper semi-continuous log-concave functions on* $\mathbb{R}^d$*. For every* $\sigma \subseteq [n] = \{1, 2, \ldots, n\}$*, let* f_σ *denote the pointwise minimum* $f_\sigma(x) = \min\{f_i(x) : i \in \sigma\}$*. Then there is a set* $\sigma \in \binom{[n]}{\leq 2d+1}$ *of at most* $2d+1$ *indices such* *that*

$$\int_{\mathbb{R}^d} f_\sigma \le c_d \int_{\mathbb{R}^d} f_{[n]},$$

holds with a universal consant $c_d > 0$.

Our recently published paper [8] has received 4 independent citations according to Google Scholar.

1.3. For a compact set $A \subset \mathbb{R}^d$ and an integer $k \ge 1$, let us denote by $A[k] = \{a_1 + \cdots + a_k : a_1, \ldots, a_k \in A\}$ the **Minkowski sum of k copies** of A. A theorem of Shapley, Folkmann and Starr (1969) states that $\frac{1}{k}A[k]$ converges to the convex hull of A in the Hausdorff distance as k tends to infinity. Bobkov, Madiman and Wang (2011) conjectured that the volume of $\frac{1}{k}A[k]$ is non-decreasing in k, or in other words, in terms of the volume deficit between the convex hull of A and $\frac{1}{k}A[k]$, this convergence is monotone. It was proved by Fradelizi, Madiman, Marsiglietti and Zvavitch (2016) that this conjecture holds if $d = 1$ but fails for any $d \ge 12$. Together with Matthieu Fradelizi and Artem Zvavitch, Zsolt Lángi [5] showed that **the conjecture by Bobkov, Madiman and Wang is true** for any star-shaped set $A \subset \mathbb{R}^d$ for $d = 2$ and $d = 3$, and also for arbitrary dimensions $d \ge 4$ under the condition $k \ge (d-1)(d-2)$. In addition, they investigated the conjecture for connected sets and presented a counterexample to a generalization of the conjecture to the Minkowski sum of possibly distinct sets in $\mathbb{R}^d$, for any $d \ge 7$.

1.4. A classical theorem of Alon and Milman [AM83] states that any d dimensional centrally symmetric convex body has a projection of dimension $m \ge e^{c\sqrt{\ln d}}$ which is either close to the m-dimensional Euclidean ball or to the m-dimensional cross-polytope.

Unlike several other fundamental results on symmetric convex bodies, this result had not been extended to non-symmetric convex bodies, until it was carried out by the PI in [20], a paper that appeared in GAFA Lecture Notes.

1.5. Volume of geometric objects plays an fundamental role in applied and theoretical mathematics, and in particular in discrete geometry. The **book titled *Volumetric Discrete Geometry*** [25] written by Károly Bezdek and Zsolt Lángi introduces problems related to recently found aspects of the volume, and discusses a variety of modern methods relying on the application of volume in geometric problems.

2. Quantitative Helly-type problems

Helly's classical theorem states that a finite family of convex sets in Euclidean d-space has non–empty intersection if, and only if, so does any subfamily of $d+1$ members. Bárány, Katchalski and Pach [BKP82; BKP84] proved a quantitative version, the **Volume Helly Theorem**: *If any $2d$ members of a finite family of convex sets in Euclidean d-space have intersection of volume at least one, then the intersection of the whole family is of volume at least $c_d > 0$, a universal constant depending only on d.*

In [BKP82], the bound $c_d \geq d^{-2d^2}$ is proved, and d^{-cd} is conjectured, which was confirmed in [Nas16].

2.1. Together with Gábor Damásdi and Viktória Földvári, MSc and PhD students at the time of the research, the PI showed a **colorful version** of the Volume Helly Theorem [14], a result with 8 independent citations on Google Scholar. As a continuation, with Attila Jung, an MSc student at the time of the research, the PI [9] found a **fractional version** of the Volume Helly Theorem, and they deduced a quantitative version of a number of classical Helly-type results, most notably, the (p, q)-theorem of Alon and Kleitman [AK92].

2.2. G. Ivanov and the PI [7] studied a problem that we may call the **Diameter Helly Theorem**, which reads as follows.

Theorem 4 (G. Ivanov, MN [7]). *Let $\{K_1, \ldots, K_n\}$ be a family of closed convex sets in $\mathbb{R}^d$ such that their intersection $K = K_1 \cap \cdots \cap K_n$ is a convex body. Then there is a selection $\sigma \in \binom{[n]}{\leq 2d}$ of at most $2d$ indices such that*

$$vol(K_\sigma) \leq (2d)^{3d} vol(K) \quad \text{and} \quad diam(K_\sigma) \leq (2d)^3 diam(K),$$

where $K_\sigma = \cap_{i\in\sigma} K_i$.

After a non-polynomial upper bound on $diam(K_\sigma)/diam(K)$ (where K_σ is the choice with the smallest diameter) was shown in [BKP82], Brazitikos [Bra18] (see also [Bra16]) established the first polynomial bound: Using a sparsification result from [BSS14] (see also [Bar14, Lemma 3.1]) related to contact points of John's ellipsoid, he showed $\frac{diam(K_\sigma)}{diam(K)} \leq cd^{11/2}$ with an absolute constant $c > 0$. We improved this result with a completely different, elementary argument, which was applied and improved by Almendra-Hernández, Ambrus and Kendall [AAK22]. We proved also a lower bound on $diam(K_\mu)/diam(K)$ of order $d^{-1/2}$, which is the order of magnitude conjectured by Bárány, Katchalski and Pach [BKP82]. Our paper [7], despite its very recent publication, has received 4 independent citations on Google Scholar.

3. Approximation

We describe a joint work [18] of Grigory Ivanov, Alexandr Polyanskii and the PI.

Rudelson's theorem [Rud99] states that if for a set of unit vectors u_i in $\mathbb{R}^d$ and positive weights c_i, we have that $\sum c_i u_i \otimes u_i$ is the identity operator I on $\mathbb{R}^d$, then the properly scaled average of a *random sample* of $Cd \ln d$ of these diadic products is close to I. The $\ln d$ term cannot be removed. The problem of approximating the average of a set of matrices as a weighted sum of a subset of the matrices is a very general and fundamental question in computational linear algebra, and is related to not only convex geometry, where Rudelson's motivation came from, but also quantum mechanics, see the celebrated paper [MSS15].

The recent result of Batson, Spielman and Srivastava [BSS14] and its improvement by Marcus, Spielman and Srivasta [MSS15] (see also [Sri12] and [FY17]) show that the $\ln d$ term can be removed, if one wants to show the *existence* of a good approximation of I

as the average of a few diadic products. It is known that essentially the same proof as Rudelson's yields a more general statement about the average of *positive semi-definite matrices.*

In [18] first, we give an example of an **average of positive semi-definite matrices** where *there is no approximation* of this average by Cd elements. Thus, the result of Batson, Spielman and Srivastava cannot be extended to this wider class of matrices.

Next, we present a stability version of Rudelson's result on positive semi-definite matrices, and thus, extend it to certain non-symmetric matrices. This yields applications to the study of the Banach–Mazur distance of convex bodies.

Finally, we show that in some cases, one needs to take a subset of the vectors of at least order d^2 to approximate the identity, which is a lower bound matching the obvious general upper bound $d^2 + 1$ that follows immediately from Carathéodory's theorem applied in the vector space of $d \times d$ matrices.

4. Coverings

4.1. The ***closest vector problem* (CVP)**, an important algorithmic question in the geometry of numbers is the following. Given a rational lattice $\Lambda = \{Ax : x \in \mathbb{Z}^n\}$, with $A \in \mathbb{Q}^{n\times n}$ and a target vector $t \in \mathbb{Q}^n$, the task is to find a vector in Λ close to t with respect to a given norm. Specifically, given some norm $\|\cdot\|_K$, a $(1+\epsilon)$-approximation to the closest vector problem, $(1+\epsilon)$-CVP$_K$, is to find a lattice vector whose distance to the target vector is at most $(1+\epsilon)$ times the minimal distance of the target to the lattice. Whenever K is the unit ball of the space ℓ_p^n for some $1 \le p \le \infty$, we denote the problem by $(1+\epsilon)$-CVP$_p$. It was shown that CVP is NP-hard for any ℓ_p norm [Emd81] and even NP-hard to approximate up to almost polynomial factors, [Aro95], [Din+03].

In [13], Moritz Venzin and the PI presented an algorithms for the $(1+\epsilon)$-approximate version of the closest vector problem for certain norms. The previously fastest algorithm (Dadush and Kun [DK16]) for general norms in dimension n has running time of $2^{O(n)}(1/\epsilon)^n$, which we improved substantially in the following two cases.

First, for ℓ_p-norms with $p > 2$ (resp. $p \in [1,2]$) fixed, we present an algorithm with a running time of $2^{O(n)}(1+1/\epsilon)^{n/2}$ (resp. $2^{O(n)}(1+1/\epsilon)^{n/p}$). This result is based on a geometric covering problem, that was introduced in the context of CVP by Eisenbrand et al. [EHN11]: *How many convex bodies are needed to cover the ball of the norm such that, if scaled by factor 2 around their centroids, each one is contained in the* $(1+\epsilon)$*-scaled homothet of the norm ball?* In [13], we provide upper bounds for this $(2,\varepsilon)$-*covering number* by exploiting the *modulus of smoothness* of the ℓ_p-balls. Applying a covering scheme, we can boost any 2-approximation algorithm for CVP to a $(1+\epsilon)$-approximation algorithm with the improved running time, either using a straightforward sampling routine or using the deterministic algorithm of Dadush for the construction of an epsilon net. Furthermore, we establish a connection between the *modulus of smoothness* of the unit ball of the norm and *lattice sparsification.* As a consequence, using the enumeration and sparsification tools developped by Dadush, Kun, Peikert and Vempala [DPV11], we present a simple alternative to the boosting procedure with the same time and space requirement for ℓ_p norms.

4.2. Together with Jana Cslovjecsek, Romanos Diogenes Malikiosis and Matthias Schymura, the PI [4] considered the problem of computing the exact value of the ***covering radiius*** of a convex polytope P in $\mathbb{R}^d$, a parameter defined with respect to a lattice Λ as the smallest non-negative real number μ such that the lattice arrangement $\mu K + \Lambda = \bigcup_{z\in\Lambda}(\mu K + z)$ of μK is a covering of $\mathbb{R}^d$, that is, $\mu K + \Lambda = \mathbb{R}^d$. As our main result, we described a new algorithm for this problem, which is simpler, more efficient and easier to implement than the only prior algorithm of Kannan [Kan92].

Motivated by a variant of the famous *Lonely Runner Conjecture*, a problem originally stated by Wills [Wil68] in the 1960's as a problem in Diophantine Approximation, we used its geometric interpretation in terms of covering radii of zonotopes, and applied our algorithm to prove the first open case of three runners with individual starting points.

5. Arrangements of convex bodies

5.1. In [12], Konrad Swanepoel and the PI studied the contact structure of ***totally separable*** **packings** of translates of a convex body K in $\mathbb{R}^d$, that is, packings where any two translates of the packing have a separating hyperplane that does not intersect the interior of any translate in the packing. The *separable Hadwiger number* $H_{\mathrm{sep}}(K)$ of K is defined to be the maximum number of translates touched by a single translate, with the maximum taken over all totally separable packings of translates of K. We showed that for each $d \geq 8$, there exists a smooth and strictly convex K in $\mathbb{R}^d$ with $H_{\mathrm{sep}}(K) > 2d$, and asymptotically, $H_{\mathrm{sep}}(K) = \Omega\left((3/\sqrt{8})^d\right)$.

We showed also that Alon's packing of Euclidean unit balls [Alo97], where each translate touches at least $2^{\sqrt{d}}$ others whenever d is a power of 4, can be adapted to give a totally separable packing of translates of the ℓ_1-unit ball with the same touching property.

We also considered the maximum number of touching pairs in a totally separable packing of n translates of any planar convex body K, and proved that the maximum equals $\lfloor 2n - 2\sqrt{n}\rfloor$ if and only if K is a quasi hexagon, thus completing the determination of this value for all planar convex bodies.

5.2. Let X be a finite subset of $\mathbb{R}^d$ whose diameter with respect to a fixed norm is one. Then X is called ***k-diametral***, if among any k elements, there is a pair at unit distance. On the other hand, independently of the norm, the set X is called *k-antipodal*, if among any k elements there is a pair whose elements lie on two distinct parallel supporting hyperplanes of X.

In [3], Károly Bezdek and Zsolt Lángi showed that the structure of k-diametral point configurations is closely related to the properties of k-antipodal point configurations in $\mathbb{R}^d$. In particular, the maximum size of k-diametral point configurations of Minkowski d-spaces is obtained for given $k \geq 2$ and $d \geq 2$ generalizing Petty's results [Pet71a] on equilateral sets in Minkowski spaces. Furthermore, bounds are derived for the maximum size of k-diametral point configurations in certain Minkowski spaces (eg., in Euclidean d-space).

6. Volumetric inequalities

In [6], Ákos G. Horváth and Zsolt Lángi investigate the properties of the **convex hull and the λ-homothetic convex hull functions** ($x \mapsto vol(K \cap (x + K))$, and $x \mapsto vol(K \cap (x + \lambda K))$, respectively) of a convex body K in $\mathbb{R}^d$-space.

A famous result of Meyer, Reisner and Schmuckenschläger [MRS93] states the following. *If $K \subset \mathbb{R}^d$ is an o-symmetric convex body with the property that the volume $vol(K \cap (x+K))$ depends only on the Minkowski norm $\|x\|_K$, then K is an ellipsoid.*

In [6], it is shown that the above statement does not hold for any convex body that in not o-symmetric. Furthermore, the equivalence of the polar projection body problem raised by Petty [Pet71b], and a conjecture of G. Horváth and Lángi about translative constant volume property of convex bodies is proved, and a short proof of some theorems of Jerónimo-Castro [Jer15] about the homothetic convex hull function is given. A homothetic variant of the translative constant volume property conjecture for 3-dimensional convex polyhedra is also considered, and the results are applied to describe properties of the illumination bodies of convex bodies.

7. Planar configurations

A convex polygon Q is circumscribed about a convex polygon P if every vertex of P lies on at least one side of Q . In [23], Markus Ausserhofer, Susanna Dann, Zsolt Lángi and Géza Tóth present an **algorithm for finding a maximum area convex polygon circumscribed** about any given convex n-gon in $O(n^3)$ time. As an application to a problem in statistics, they disprove a conjecture of Farris [Far10].

8. Students

Both Zsolt Lángi and the PI worked with MSc. and Ph.D. students during the project, who got engaged in it. *Sami Almohammad* is a Ph.D student at ELTE supervised by the PI and co-supervised by Lángi. He is about to defend his thesis, part of which is based on a joint work [2] of three of us. *Bushra Basit* is a Ph.D. student and Budapest University of Technology and Economics (BUTE) with Lángi's supervision, they co-authored [11]. *Nóra Frankl*, *Viktória Földvári*, *Gábor Damásdi* and *Attila Jung* wrote their MSc. theses at ELTE with the PI's supervision, part of their research is presented in [30; 14; 9]. *Lili Ködön* wrote her BSc. thesis at BUTE under Lángi's supervision, and [10] is their joint work. *Máté Kadlicskó* is about to defend his MSc. thesis at BUTE, also with Lángi's supervision, their joint work [1] is the most recent output of the project.

Nóra Frankl and Viktória Földvári have since defended their Ph.D. theses at LSE, London and at ELTE, respectively, while Gábor Damásdi is about to defend his at ELTE. Attila Jung has just started his Ph.D. program at ELTE.

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