OTKA 2016-22 closing report

Our team published about 100 works in the area of the project, that is, analytic and combinatorial number theory. During the project the greatest distinction was obtained by Endre Szemerédi who obtained in 2020 the Saint Stephen Order the highest distinction in Hungary from the head of state of Hungary. He gave also a distinguished lecture at the Congress of EMS, the Abel lecture in 2016 (which was published later in the volume of the Congress). Gergely Harcos, a member of our team, won a Lendület project of the Hungarian Academy of Sciences .

The results of the team appeared in English in international journals and proceedings of international conferences. We gave a more detailed account of them in earlier reports, so now we try to summarize just some of the most important developments of the project.

The PI continued his earlier investigations about gaps between primes, an approximation to the 2300 years old Twin Prime Conjecture. The base for the present methods was created by him, in collaboration with Goldston and Yildirim showing that there exist infinitely many prime gaps $d_n=p_{(n+1)}-p_n$ which normalized that means divided by their expected value logn take arbitrarily small values. Later their method was improved by Zhang, Maynard and Tao (unpublished) which in the form given by Maynard made possible the investigation of several consecutive prime gaps as well. Erdős conjectured more than 60 years ago that these normalized primegaps are everywhere dense in the nonnegative real line. As an approximation to this Banks, Freiberg and Maynard showed recently that at least 1/8th of all real nonnegative numbers below a large number T appear as limit points of the sequence $d_n/\log n$. This was improved to one quarter of the non-negative real numbers by the PI. Another problem was raised by Erdős, Pólya and Turán 70 years ago. In 1948 Erdős and Turán showed that $d_{n+1}-d_n$ changes sign infinitely often and together with Pólya asked for a necessary and sufficient condition that a fixed linear combination of $r>1$ consecutive prime gaps d_{n+i} $i=(1,2,...r)$ should change sign infinitely often as n runs through the sequence of all natural numbers. They conjectured that the condition is that the non-zero values among the coefficients of the linear combination cannot all have the same sign but they could prove only much weaker relations. This conjecture was also proved by the PI using the Maynard-Tao method and other important ideas of the above mentioned recent work of W.D. Banks, T. Freiberg and J.Maynard.

Concerning the question of distribution of primegaps we succeeded to show a far reaching common generalization of two very important breakthroughs in prime number theory: the result of Green and Tao according to which there are arbitrarily long arithmetic progressions in the sequence of primes and the result of Maynard and Tao, according to which there are infinitely many non overlapping intervals of length C(r) - a constant depending exponentially on r such that we have a block of r consecutive primes in the given bounded intervals. Our very general theorem asserts that for every given r and for a positive proportion of all r-tuples $H_r= \{h_1, h_2,...h_r\}$ we have for every k infinitely many k-term arithmetic progressions $\{n_1, n_2,... n_k\}$ with the property that all numbers of the form n_i+h_j $(i=1,2,...k\}$, $j=\{1,2,...r\})$ are primes.

We continued our investigations on the application of an approximate formula for Goldbach's Problem, developed by us. I.Z. Ruzsa and the PI showed that this method has applications on the Goldbach problem in sparse sequences and in the famous Linnik-Goldbach problem. Linnik proved seventy years ago that every even number can be written as the sum of two primes and a bounded number of powers of two. His theorem provided no explicit results about the needed number K of powers of two. Nonetheless it showed (with a non precise formulation) that every integer can be converted into a Goldbach number (that is a number being the sum of two primes) if we allow to change a bounded number K of its binary digits, independently from the size of the number. The first unconditional explicit value of K was around 50,000 for large enough integers, proved at the end of the last century. Now we showed that this theorem is true with K=8. That is, the following approximation of the Goldbach problem is true: every sufficiently large even number can be written as the sum of two primes and eight powers of two.

Automorphic forms are harmonic waves with a rich symmetry, which help us to understand the whole numbers. An important task is to study the value distribution of automorphic forms. We have investigated in depth the global maximum of cusp forms defined on nxn matrix groups, and where the maximum can occur within the matrix group. For n=3 we gave a concrete upper bound in terms of the Laplace eigenvalue], while for general n we proved that the maximum can be bounded from above by a power of the Laplace eigenvalue in which the exponent is a cubic polynomial of n. In the other direction, Brumley–Templier (2014) established similar lower bounds. We established bounds that reproduced or improved upon all known special cases, not only in terms of the Laplace eigenvalue but also in terms of the level of the cusp form. In particular, our works extend the celebrated work of Iwaniec-Sarnak (1995). In our papers the cusp forms are spherical, meaning that they are invariant under the maximal compact subgroup of the underlying matrix group. We initiated a new line of research by proving strong bounds for the restricted maximum of non-spherical cusp forms on the group of 2x2 complex unimodular matrices. On the way, we developed analytic theory of independent interest, including uniform strong localization estimates for generalized spherical functions and a Paley-Wiener theorem for the corresponding spherical transform acting on the space of rapidly decreasing functions. The lengths of primitive closed geodesics on the modular surface are distributed similarly as the prime numbers. This fact is expressed by the prime geodesic theorem of Huber (1959), and automorphic forms play a key role in its proof. We gave a strong upper bound for the square mean of the error term in the prime geodesic theorem. We made progress on the following problem in extremal graph theory: how many Hamiltonian paths can be given in the complete graph on n vertices such the union of any two contains a cycle of length 2k. We provide an upper bound that matches (essentially) the lower bound when k=2, and improves on the previously known upper bounds also when k>3. Our arguments utilize expander graphs and advanced results in prime number theory. Another work of us has close ties with the classical theorems of Minkowski (1891). We estimate, in a number field, the number of elements and the maximal number of linearly independent elements, with prescribed bounds on their valuations. As a by-product, we obtain new bounds for the successive minima of ideal lattices. Our arguments combine group theory, ramification theory, and the geometry of numbers.

András Biró solved effectively the class number one problem for a certain family of real quadratic fields. This family contains the so-called Yokoi discriminants, so our result is a generalization of the solution of Yokoi’s Conjecture. But this family may contain also infinitely many fields with comparatively larger fundamental units than the fields in the Yokoi family. The proof is also a generalization of the proof of Yokoi’s Conjecture. These problems originated in the works of Gauss more then twohundred years ago about the class number of quadratic fields.

Szilárd Révész devoted five long papers to the study of so called Beurling primes, in which he proves analogues of classical results. The works make an important step towards the further development of the theory of Beurling primes. They show analogues of the Riemann - von Mangoldt formula, Carlson's density theorem, and results of Ingham, Turán, Révész and the PI about the connection of the distribution of zeros of the relevant zeta-function and the oscillation of the remainder term of the prime number theorem.

András Sárközy and his coauthors (in particular Katalin Gyarmati, a member of our team) introduced about 20 years ago years ago new quantitative measures of pseudorandom binary sequences. These are important in cryptography and therefore have a wide range of applications. For example, they introduced the measure of irregularities of distribution relative to short arithmetic progressions. In one paper they give constructive bounds for the minimal value of this measure for binary sequences of a given length In another work they showed that finite binary sequences possessing good pseudorandom properties in terms of these new measures usually also pass most of the so called NIST tests. In another work they proved that most of these constructions can be also adapted for constructing quasi- random graphs

In a long series of works Endre Szemerédi and his coauthors proved a weaker form of the Loebl-Komlós-Sós conjecture according to which every graph on n vertices of which at least n/2 have degrees k, contains as subgraphs all trees with at most k edges. The proof is contained in five subsequent papers of a total of more than 200 pages. It is proved in the series of papers that for every c>0 there exists a number k_0 such that for every k>k_0 every n-vertex graph G with at least (1/2+c) vertices of degree at least (1+c)k contains each tree T of order k as a subgraph. The Brown-Erdős-Sós conjecture, one of the central conjectures in extremal combinatorics, states that for any integer m≥6, if a 3-uniform hypergraph on n vertices contains no m vertices spanning at least m−3 edges, then the number of edges is o(n^2). Endre Szemerédi proved the conjecture for triple systems coming from finite abelian groups.