FINAL REPORT FOR NKFIH GRANT PROPOSAL NO. 118946: BOUNDS FOR EIGENFORMS ON ARITHMETIC MANIFOLDS

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This report summarizes the results of a research group supported by an 18-month focused "ERC Helper" grant of the NKFIH (National Research, Development and Innovation Office). The group consisted of Gergely Harcos (PI), András Biró, Péter Maga, and Árpád Tóth. The work was carried out at the Alfréd Rényi Institute of Mathematics and at Eötvös Loránd University. In the meantime, the PI successfully applied for a 5-year "Lendület" grant of the Hungarian Academy of Sciences, which allows the group to further grow professionally and physically (currently there are 7 members).

The new results appear in 9 papers spanning 5 topics: bounds for automorphic forms [BHMM, BHM1, BHM2], modular invariants [DIT1, DIT2, HIPT], the hyperbolic circle problem [Bi4], hypergeometric functions [Bi3], and random power series [MaMa]. A detailed account is given below.

1. Bounds for automorphic forms

1.1. **Results for the group** GL_2**.** We proved a strong and natural upper bound for the global sup-norm of an L^2-normalized spherical Hecke–Maaß cuspidal newform ϕ on $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$, where F is a number field with adele ring $\mathbb{A}$. It extends the state-of-the-art results for the rational field $F=\mathbb{Q}$ by Templier [Te] and for the Gaussian field $F=\mathbb{Q}(i)$ by Blomer–Harcos–Milićević [BHM3]. To formulate the result, we note that the level of ϕ is an ideal $\mathfrak{n}$ in the ring of integers $\mathfrak{o}$ of F, and we shall abbreviate the norm $[\mathfrak{o}:\mathfrak{n}]$ by $|\mathfrak{n}|$. We also associate to ϕ the tuple $\lambda:=(\lambda_1,\dots,\lambda_r,\lambda_{r+1},\dots,\lambda_{r+s})$ of its Laplace eigenvalues at the r real places and the s complex places of F, and we write $|\lambda|_\infty:=\prod_{j=1}^r\lambda_j\prod_{j=r+1}^{r+s}\lambda_j^2$.

Theorem 1 ([BHMM])**.** *Assume that the level* $\mathfrak{n}$ *is square-free. Then* $\|\phi\|_\infty\ll_{F,\varepsilon}|\lambda|_\infty^{5/24+\varepsilon}|\mathfrak{n}|^{1/3+\varepsilon}$ *holds for any* $\varepsilon>0$ *when* F *is totally real, and* $\|\phi\|_\infty\ll_{F,\varepsilon}|\lambda|_\infty^{5/24+\varepsilon}|\mathfrak{n}|^{5/12+\varepsilon}$ *holds when* F *is a CM-field.*

We note that the "trivial bound" would read $\|\phi\|_\infty\ll_F|\lambda|_\infty^{1/4}|\mathfrak{n}|^{1/2}$, and in fact the above bounds are special cases of more general nontrivial bounds valid for all number fields. To formulate these, we write $|\lambda|_{\mathbb{R}}:=\prod_{j=1}^r\lambda_j$ and $|\lambda|_{\mathbb{C}}:=\prod_{j=r+1}^{r+s}\lambda_j^2$.

Theorem 2 ([BHMM])**.** *Assume that the level* $\mathfrak{n}$ *is square-free. Then for any* $\varepsilon>0$*, we have*

$$\|\phi\|_\infty\ll_{F,\varepsilon}|\lambda|_\infty^{5/24+\varepsilon}|\mathfrak{n}|^{1/3+\varepsilon}+|\lambda|_{\mathbb{R}}^{1/8+\varepsilon}|\lambda|_{\mathbb{C}}^{1/4+\varepsilon}|\mathfrak{n}|^{1/4+\varepsilon}.$$

Furthermore, if F *is not totally real and* $m\geqslant 2$ *denotes the degree of* F *over its maximal totally real subfield, then for any* $\varepsilon>0$*, we have*

$$\|\phi\|_\infty\ll_{F,\varepsilon}|\lambda|_\infty^{(4m-3)/(16m-8)+\varepsilon}|\mathfrak{n}|^{(4m-3)/(8m-4)+\varepsilon}.$$

Very recently, Assing [As] generalized the above bounds to arbitrary level and arbitrary nebentypus, by combining our ideas (which are of arithmetic, geometric, and combinatorial in nature) with the more representation theoretic ideas of Saha [Sa].

1.2. **Results for the group** GL_n**.** We proved an explicit upper bound for the global sup-norm of a Maaß cusp form ϕ on the space $X_n:=\mathrm{GL}_n(\mathbb{Z})\backslash\mathscr{H}_n$, where $\mathscr{H}_n:=\mathrm{Z}_n(\mathbb{R})\backslash\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$ is the usual higher rank generalization of the upper half-plane. It is the first result of this kind in the literature: the earlier results were concerned with the local sup-norm restricted to a compact subset (see [BlMa, Mar]), or with lower bounds for the global sup-norm (see [BrTe]).

Theorem 3 ([BHM2])**.** *Let ϕ be an L^2-normalized Maaß cusp form on X_n with Laplace eigenvalue λ_ϕ. Then for any $\varepsilon > 0$, we have $\|\phi\|_\infty \ll_{n,\varepsilon} \lambda_\phi^{(n^2-2)(n+1)/16+\varepsilon}$.*

This result should be compared with the lower bound $\|\phi\|_\infty \gg_{n,\varepsilon} \lambda_\phi^{n(n-1)(n-2)/24-\varepsilon}$ established under some mild hypotheses by Brumley and Templier [BrTe, Thm. 1.1]. We note that for n of moderate size, improvements are possible. In particular, for $n = 2$ and $n = 3$, the best known upper bounds for $\|\phi\|_\infty$ appear in the seminal paper of Iwaniec and Sarnak [IwSa] and in our companion paper [BHM1].

Theorem 3 is a consequence of two more refined bounds that follow from the Fourier–Whittaker expansion and the Selberg pre-trace formula combined with ideas from the geometry of numbers. To formulate these, we recall that X_n has a fundamental domain lying in the standard Siegel set

$$|x_{ij}| \leqslant 1/2 \ \text{ for } \ j > i \qquad \text{and} \qquad y_1, \dots, y_{n-1} \geqslant \sqrt{3}/2, \tag{1}$$

with coordinates on $\mathscr{H}_n$ as in [Go, Def. 1.2.3]. The first bound below is useful when the y_i's are large on average, while the second bound below is useful in the opposite case.

Theorem 4 ([BHM2])**.** *Let ϕ be an L^2-normalized Maaß cusp form on X_n, and let $z \in \mathscr{H}_n$ be a point in the Siegel set (1). There exists a constant $c_n > 0$ such that*

$$\phi(z) \ll_n \lambda_\phi^{n^3} \exp\left(-c_n \mathscr{Y}(z)/T_\mu\right),$$

where

$$\mathscr{Y}(z) := \max_{1 \leqslant j \leqslant n-1} \max\left(\prod_{i=1}^{j} y_i^{j-i+1}, \prod_{i=1}^{j} y_{n-i}^{j-i+1}\right)^{\frac{2}{j(j+1)}}.$$

In addition, we have

$$\phi(z) \ll_n \lambda_\phi^{(n^2-n)/8} + \lambda_\phi^{(n^2-n-1)/8} \prod_{i=1}^{n-1} y_i^{i(n-i)/2}.$$

In order to control the terms in the Fourier–Whittaker expansion of the cusp form ϕ, we derived uniform upper bounds for the underlying Jacquet–Whittaker function $\mathscr{W}_\mu$, which are of independent interest. For the sake of exposition, we restrict here to the tempered case (which by the generalized Ramanujan–Selberg conjecture should always be the case). Then, the corresponding archimedean Langlands parameters are purely imaginary:

$$\mu = (\mu_1, \dots, \mu_n) \in (i\mathbb{R})^n, \qquad \mu_1 + \dots + \mu_n = 0, \qquad \Im\mu_1 \geqslant \ldots \geqslant \Im\mu_n. \tag{2}$$

With this notation, the Laplace eigenvalue equals

$$\lambda_\phi = \frac{n^3 - n}{24} - \frac{\mu_1^2 + \dots + \mu_n^2}{2},$$

and it is convenient to introduce

$$T_\mu := \max(2, |\mu_1|, \dots, |\mu_n|) \asymp_n \lambda_\phi^{1/2}.$$

Theorem 5 ([BHM2])**.** *Let $t = \operatorname{diag}(t_1, \dots, t_n) \in \mathrm{GL}_n(\mathbb{R})$ with $t_1, \dots, t_n > 0$. Assume that the Langlands parameters satisfy (2). Then for any $\varepsilon > 0$, we have*

$$\mathscr{W}_\mu(t) \ll_{n,\varepsilon} \left(\prod_{j=1}^{n} t_j^{n+1-2j}\right)^{1/2-\varepsilon} \exp\left(-\frac{1}{T_\mu} \sum_{j=1}^{n-1} \frac{t_j}{t_{j+1}}\right).$$

Our actual result is stronger and addresses the non-tempered case as well, but we preferred the above clean formulation. In particular, $\left\|\mathscr{W}_\mu\right\|_\infty \ll_n T_\mu^{(n^3-n)/12}$, which complements the analogous lower bound $\left\|\mathscr{W}_\mu\right\|_\infty \gg_n T_\mu^{n(n-1)(n-2)/12}$ established under some mild hypotheses by Brumley and Templier [BrTe, Thm. 1.4]. Our companion paper [BHM1] contains some more refined bounds for the case $n = 3$.

2. Modular invariants

2.1. Geometric invariants for real quadratic fields. In [DIT1] we introduced a new geometric invariant associated to a narrow ideal class of a real quadratic field. This invariant is a finite area hyperbolic surface with a boundary that maps naturally on the modular surface. The boundary is a simple closed geodesic whose image in the modular surface is the usual modular closed geodesic associated to the ideal class. Its length is well known to be expressible in terms of a fundamental unit of the field. The area of the surface is determined by the length of an associated minus (or backward) continued fraction. The surface contains more information than the closed geodesic alone. We obtained a result about the distribution properties of the surface as it lies in the modular surface.

Theorem 6 ([DIT1]). *Suppose that for each positive fundamental discriminant $D > 1$ we choose a genus $G_D \in \mathrm{Gen}(K)$. Let Ω be an open disc contained in the fundamental domain $\mathscr{F}$ for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and let $\Gamma\Omega$ be its orbit under the action of Γ. We have*

$$\frac{\pi}{3} \sum_{A \in G_D} \mathrm{area}(\mathscr{F}_A \cap \Gamma\Omega) \sim \mathrm{area}(\Omega) \sum_{A \in G_D} \mathrm{area}(\mathscr{F}_A), \tag{3}$$

as $D \to \infty$ through fundamental discriminants.

This problem is closely allied with (and in fact completes in a natural way) the problem of the uniform distribution of the closed geodesics on the modular surface when ordered by their associated discriminant [Du]. The analytic approach to the closed geodesic problem leads to estimating the Fourier coefficients of Maaß cusp forms of weight $1/2$. For the surface problem, this approach also leads to estimating these Fourier coefficients (for different indices), but it requires interesting and nontrivial extensions of formulas of Hecke and Katok–Sarnak [KaSa]:

Theorem 7 ([DIT1]). *Let*

$$\varphi(z) = 2y^{1/2} \sum_{n \neq 0} a(n) K_{ir}(2\pi|n|y) e(nx)$$

be a fixed even Hecke–Maaß cusp form for Γ. Then there exists a unique nonzero $F(z)$ with weight $1/2$ for $\Gamma_0(4)$ with Fourier expansion

$$F(z) = \sum_{\substack{n \equiv 0,1 \,(\mathrm{mod}\, 4) \\ n \neq 0}} b(n) W_{\frac{1}{4}\mathrm{sgn}\, n, \frac{ir}{2}}(4\pi|n|y) e(nx),$$

such that for any pair of coprime fundamental discriminants d' and d we have

$$12\sqrt{\pi}|D|^{\frac{3}{4}} b(d')\overline{b(d)} = \langle \varphi, \varphi \rangle^{-1} \sum_{A \in \mathrm{Cl}^+(K)} \chi(A) \begin{cases} \frac{\lambda}{2} \int_{\mathscr{F}_A} \varphi(z)\, d\mu & \text{if } d', d < 0 \\ \int_{\partial \mathscr{F}_A} \varphi(z) y^{-1} |dz| & \text{if } d', d > 0 \\ 2\sqrt{\pi}\, \omega_D^{-1}\, \varphi(z_A) & \text{if } d'd < 0, \end{cases}$$

where χ is the genus character associated to $D = d'd$. Here $\langle F, F \rangle = \int_{\Gamma_0(4)\backslash\mathscr{H}} |F|^2 d\mu = 1$, and the value of $b(n)$ for a general discriminant $n = dm^2$ for $m \in \mathbb{Z}^+$ is determined by means of the Shimura relation

$$m \sum_{\substack{n|m \\ n>0}} n^{-\frac{3}{2}} \left(\frac{d}{n}\right) b\left(\frac{m^2 d}{n^2}\right) = a(m) b(d).$$

2.2. Modular cocycles and linking numbers. The 3-manifold $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is diffeomorphic to the complement of the trefoil knot in S^3. Ghys [Gh] showed that the linking number of this trefoil knot with a modular knot is given by the Rademacher symbol, which is a homogenization of the classical Dedekind symbol. The Dedekind symbol arose historically in the transformation formula of the logarithm of Dedekind's eta function under $\mathrm{SL}_2(\mathbb{Z})$. In [DIT2] we gave a generalization of the Dedekind symbol associated to a fixed modular knot.

Let $\mathscr{C}$ be a conjugacy class of a hyperbolic element $\sigma \in \Gamma$. Define the weight two 1-cocycle for $c \neq 0$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by

$$r_{\mathscr{C}}(\gamma, z) := \varepsilon_{\mathscr{C}} \sum \left(\frac{1}{z-w} - \frac{1}{z-w'} \right), \tag{4}$$

where the sum is over the fixed points w', w of $\sigma \in \mathscr{C}$, satisfying $w' < -d/c < w$ and

$$\varepsilon_{\mathscr{C}} := \begin{cases} 1 & \text{if } \sigma \not\sim \sigma^{-1} \\ 2 & \text{if } \sigma \sim \sigma^{-1} \end{cases}. \tag{5}$$

If $c = 0$ we let $r_{\mathscr{C}}(\gamma, z) = 0$. We then have

Theorem 8 ([DIT2])**.** *Let $r_{\mathscr{C}}(\gamma, z)$ be defined as in (4). Then $r_{\mathscr{C}}(\gamma, z)$ is a weight 2 cocycle for Γ.*

Let $R_{\mathscr{C}}(\gamma, z)$ be the unique primitive of $r_{\mathscr{C}}(\gamma, z)$. Define the Dedekind symbol for $\mathscr{C}$ and $\gamma \in \Gamma$ by

$$\Phi_{\mathscr{C}}(\gamma) := \tfrac{2}{\pi} \lim_{y \to \infty} \Im R_{\mathscr{C}}(\gamma, iy). \tag{6}$$

Theorem 9 ([DIT2])**.** *$\Phi_{\mathscr{C}}(\gamma)$ exists and is an integer.*

This symbol also arises in the transformation formula of a certain modular function. The homogenization of this symbol, which generalizes the Rademacher symbol, gives the linking number between two distinct symmetric links formed from modular knots.

Theorem 10 ([DIT2])**.** *Let $\mathscr{C}_\sigma$ and $\mathscr{C}_\gamma$ denote the links associated to two different primitive conjugacy classes, and let $\Psi_{\mathscr{C}_\sigma}(\gamma) := \lim_{n\to\infty} \Phi_{\mathscr{C}_\sigma}(\gamma^n)/n$. Then $Lk(\mathscr{C}_\sigma, \mathscr{C}_\gamma) = \Psi_{\mathscr{C}_\sigma}(\gamma)$.*

It can be computed in terms of a special value of a certain Dirichlet series

$$L_{\mathscr{C}}(s, \alpha) = \sum_{n \geqslant 1} a_{\mathscr{C}}(n) e(n\alpha) n^{-s}, \tag{7}$$

where the coefficient $a_{\mathscr{C}}(n)$ is given by the cycle integral of a certain Γ-invariant function along the closed geodesic associated to $\mathscr{C}$.

Theorem 11 ([DIT2])**.** *Let $L_{\mathscr{C}}(s, a/c)$ be the Dirichlet series as in (7). Then $L_{\mathscr{C}}(s, a/c)$ converges for $\Re(s) > 9/4$, has a meromorphic continuation to $s > 0$ and is holomorphic at $s = 1$. Moreover,*

$$\Phi_{\mathscr{C}}(\gamma) = -\tfrac{1}{\pi^2} \Re L_{\mathscr{C}}(1, a/c). \tag{8}$$

This symbol also satisfies the following reciprocity law.

Theorem 12 ([DIT2])**.** *Assume that $(a, c) = 1$ and $ac \neq 0$. Then*

$$\frac{1}{i\pi}\left[L_{\mathscr{C}}(1, a/c) - L_{\mathscr{C}}(1, -c/a)\right] = -2\left(\tfrac{a^2+c^2+1}{ac}\right)\log\lambda - \nu_{\mathscr{C}}(a/c), \tag{9}$$

where

$$\nu_{\mathscr{C}}(x) := \varepsilon_{\mathscr{C}} \sum_{w' < 0 < w} \left[\log\left(\frac{x-w}{x-w'} \right) - \log\left(\frac{1+xw}{1+xw'} \right) \right].$$

Here we interpret the imaginary part of the logarithm of a negative real number to be π. This result is analogous to the reciprocity law of the Dedekind symbol, and it enables a fast calculation of $L_{\mathscr{C}}(1, a/c)$ and hence also of $\Phi_{\mathscr{C}}(\gamma)$.

2.3. **A Jensen–Rohrlich type formula for the hyperbolic 3-space.** In [HIPT] we gave a Jensen–Rohrlich type formula for a certain class of automorphic functions on the hyperbolic 3-space $\mathscr{H}^3$ for the group $\mathrm{PSL}_2(\mathfrak{o}_K)$. The classical Jensen formula is a well-known theorem of complex analysis which characterizes, for a meromorphic function f on the unit disc, the value of the integral of $\log|f(z)|$ on the unit circle in terms of the zeros and poles of f inside the unit disc. An important theorem of Rohrlich [Ro] establishes a version of Jensen's formula for modular functions f with respect to the full modular group $\mathrm{PSL}_2(\mathbb{Z})$ and expresses the integral of $\log|f(z)|$ over a fundamental domain in terms of special values of Dedekind's eta function.

Consider the class $\mathscr{A}$ of functions $F:\Gamma\backslash\mathscr{H}^3\to\mathbb{R}\cup\{\infty\}$ that are square integrable and harmonic except for finitely many points $Q_1,\dots,Q_m\in X$. The behavior at these points is $F(P)=c_\ell\dfrac{\nu(Q_\ell)r_\ell}{\|P-Q_\ell\|}+O(1)$, where the constants $c_1,\dots,c_m\in\mathbb{R}$ satisfy $\sum_{\ell=1}^m c_\ell=0$.

The main theorem is the following analogue of Rohrlich's formula in this case.

Theorem 13 ([HIPT]). *Let $F:\mathscr{H}^3\to\mathbb{R}\cup\{\infty\}$ be in the class $\mathscr{A}$ described above. Then $F(\infty):=\lim_{r\to\infty}F(P)$ exists, and we have the equality*

$$\frac{1}{\mathrm{vol}(X)}\int_X F(P)\,d\mu(P)=F(\infty)+\frac{2\pi}{\mathrm{vol}(X)}\sum_{\ell=1}^m c_\ell\log\big(\eta_\infty(Q_\ell)\,r_\ell\big).$$

3. The hyperbolic circle problem

We regard the upper half-plane $\mathscr{H}$ as a model of the hyperbolic plane, and we identify its group of orientation preserving isometries with $\mathrm{PSL}_2(\mathbb{R})$ via the usual action $z\mapsto\frac{az+b}{cz+d}$ of $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{PSL}_2(\mathbb{R})$. For a discrete subgroup $\Gamma\leqslant\mathrm{PSL}_2(\mathbb{R})$, the *hyperbolic circle problem* concerns the cardinality of the finite intersection of an orbit Γz with a hyperbolic disk. Taking z as the center of the disk, in familiar notation the problem concerns

$$N(z,X):=\#\left\{\gamma\in\Gamma:\ \frac{|\gamma z-z|^2}{\Im(\gamma z)\Im(z)}+2\leqslant X\right\}.$$

Selberg understood the spectral nature of the problem, and in this way he proved (see [Iw, Theorem 12.1]) the approximation $N(z,X)=M(z,X)+O_{z,\Gamma}(X^{2/3})$, where

$$M(z,X):=\sqrt{\pi}\sum_{\substack{\frac12<s_j\leqslant1\\ \lambda_j=s_j(1-s_j)}}\frac{\Gamma\left(s_j-\frac12\right)}{\Gamma\left(s_j+1\right)}\left|\phi_j(z)\right|^2X^{s_j}.$$

Here, (ϕ_j) is a complete orthogonal system of Maaß forms on $\Gamma\backslash\mathscr{H}$ with corresponding Laplace eigenvalues (λ_j). In particular, for $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$, we have $N(z,X)=3X+O_z(X^{2/3})$. The exponent $2/3$ has never been improved for any group Γ, although it is conjectured that $1/2+\varepsilon$ is admissible (which would be close to optimal). Recently, we managed to improve the exponent on average.

Theorem 14 ([Bi4]). *For any $f\in C_c^\infty(\mathscr{H})$ we have, writing $z=x+iy$,*

$$\int_{\Gamma\backslash\mathscr{H}}f(z)\,\big(N(z,X)-M(z,X)\big)\,\frac{dxdy}{y^2}=O_{f,\Gamma,\varepsilon}(X^{5/8+\varepsilon}).$$

The main tool in the proof is our generalization of the Selberg trace formula proved earlier in [Bi1]. We note that for the special case of $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$, Petridis–Risager [PeRi] independently achieved this result with exponent $7/12+\varepsilon$ on the right hand side.

4. Hypergeometric functions

We considered a certain definite integral involving the product of two classical hypergeometric functions having complicated arguments. We showed the surprising fact that this integral does not depend on the parameters of the hypergeometric functions.

Let us write $F(a,b;c;z)$ for the Gauss hypergeometric function.

Theorem 15 ([Bi3]). *Let $0<T<S<1$, and let t be any complex number. Then*

$$\int_T^S \frac{F\left(2it,-2it;\frac{1}{2};\frac{(1+\sqrt{z})(\sqrt{z}-\sqrt{T})}{2(1-\sqrt{T})\sqrt{z}}\right)F\left(it,-it;\frac{1}{2};-\frac{(S-z)(1-z)}{(1-\sqrt{S})^2 z}\right)}{(1-z)\sqrt{z-T}\sqrt{S-z}}\,dz=\frac{\pi}{\sqrt{1-T}\sqrt{1-S}}.$$

This identity is interesting on its own right, but we stress that we discovered it while studying the special functions occurring in [Bi2], where we proved a Poisson-type summation formula with automorphic weights. We note also that the integral operator

$$g(S):=\int_0^S \frac{F\left(it,-it;\frac{1}{2};-\frac{(S-z)(1-z)}{(1-\sqrt{S})^2 z}\right)}{\sqrt{S-z}}f(z)\,dz,$$

where f and g are functions on $(0,1)$, can be inverted using Theorem 15.

5. Random power series

In our paper [MaMa], we considered the boundary behavior of the real power series $\sum_{n=1}^{\infty} a_n x^n$, where the coefficients a_n are chosen independently at random from a finite set D (of cardinality at least 2) with uniform distribution. The complex variant of the question was thoroughly examined, and it was shown that random power series in the complex plane tend to behave rather chaotically near the radius of convergence (see [St, BrSi]). We verified that the real case has similar properties, namely we proved that if the expected value of the coefficients is positive (resp. negative), then

$$\lim_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=\infty \qquad \left(\text{resp. } \lim_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=-\infty\right)$$

with probability 1. Also, if the expected value of the coefficients is 0, then

$$\limsup_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=\infty, \qquad \liminf_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=-\infty$$

with probability 1.

We investigated the analogous question in terms of Baire categories: we have shown that if the configuration space consists of all the possible choices of coefficients from D equipped with the product topology, then if each element of D is nonnegative (resp. nonpositive), then

$$\lim_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=\infty \qquad \left(\text{resp. } \lim_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=-\infty\right)$$

in a residual set. Otherwise we have

$$\limsup_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=\infty, \qquad \liminf_{x\to 1-}\sum_{n=1}^{\infty} a_n x^n=-\infty$$

in a residual set.

A motivation for this project was answering a question in [KPP]. Namely, it is a simple consequence via Bolzano's theorem that if $\sum_{d\in D} d=0$, then for almost all and residually many sequences of coefficients (a_n) the following holds. For any real number y, there are infinitely many numbers $0<x<1$ satisfying $y=\sum_{n=1}^{\infty} a_n x^n$.

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