Stochastic and Analytic Geometry
NKFIH K 116451
Report

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(Gergely Ambrus and Ferenc Fodor left the project towards the end because they joined some other projects. Their departure was approved by NKFIH)

The main focus of the project is the the so called L_p Minkowski problem and its dual version. In addition, many other problems in Stochastic and Analytic Geometry have been solved. Many of the papers resulting from the NKFIH K 11645 project have appeared in top mathematical journals like Journal of EMS, Advances in Mathematics, Journal of Differential Equations, Journal of Functional Analysis, Transactions of AMS, etc.

1 The L_p Minkowski problem and its dual

The L_p Minkowski problem for $p \in \mathbb{R}$, a Monge-Ampere equation on the sphere S^{n-1}, was posed by Lutwak around 1993, and many important results have been obtained by Chou&Wang, Guan&Lin, Hug&Lutwak&Yang&Zhang up to 2010. We note that the case $p = 1$ is the classical Minkowski problem, the case $p = 0$ is about the so-called cone volume measure introduced by Gromov and Milman around 1985, and the the case $p = -n$ is related to the so called centro-affine surface area. The L_p Minkowski problem for $p > 1$ has been mostly clarified using the technics from the classical Brunn-Minkowski Theory in the papers mentioned above. The L_p Minkowski theory for $p < 1$ received a new impetus by the 2013 paper Boroczky, Lutwak, Yang and Zhang solving the even L_0 Minkowski problem; or in words, characterizing even cone volume measures by the so called "subspace concentration property".

Turning to the results achieved within the NKFIH project K 116451, we showed that the cone volume measure satifies the "subspace concentration property" not only for origin symmetric convex bodies (the case of even measures above), but for any convex body provided the centroid is the origin[1].

The "subspace concentration property" of a cone volume measure involves a characterization of the case when the original convex body is the direct sum of two lower dimensional convex compact sets in terms of the cone volume measre. We managed to obtain a stability version of this characterization, and also providing a stability version of some fundamental inequalities for convex bodies[2].

[1] K.J. Böröczky, M. Henk: Cone-volume measure of general centered convex bodies. Advances Math., 286 (2016), 703-721.

[2] K.J. Böröczky, M. Henk: Cone-volume measure and stability. Advances in Mathematics, 306 (2017), 24-50.

If $0 < p < 1$, then we completely solved the planar version of the L_p Minkowski problem[3]. Somewhat later and independently, Chen&Li&Zhu solved the most interesting part of the L_p Minkowski problem for $0 < p < 1$ in any dimension. We managed to strengthen all known results about the L_p-Minkowski problem for $0 < p < 1$ and $-n < p < 0$[4].

If $p > 0$, then also the Orlicz versions of L_p-Minkowski problem have been solved. We solved the Orlicz versions of L_p-Minkowski problem for $-n < p < 0$[5].

The qth dual curvature measures on the sphere S^{n-1} for $q \in \mathbb{R}$ have been introduced by the recent Acta Mathematica paper by Huang, Lutwak, Yang, Zhang. Here the case $q = n$ corresponds to the cone volume measure, and the case $q = 0$ corresponds to Alexandrov's integral curvature measure, which was characterized by Alexandrov himself. If $0 < q \leq 1$, then the qth dual curvature measure was already characterized by Huang, Lutwak, Yang, Zhang. First we showed that if $1 < q < n$, then an even qth dual curvature measure satisfies the "qth subspace concentration property"[6]. Later we actually proved that the "qth subspace concentration property" characterizes even qth dual curvature measures[7].

A common generalization of the dual curvature measures and the L_p surface area measures are the L_p qth dual curvature measures, which have been introduced by Lutwak, Yang and Zhang. These new measures unifiy several other geometric measures of the Brunn-Minkowski theory and the dual Brunn-Minkowski theory. In particular, the L_p curvature measures arise from qth dual inrinsic volumes by means of Alexandrov-type variational formulas. Lutwak, Yang and Zhang also formulated the L_p dual Minkowski problem which concerns the characterization of L_p qth dual curvature measures for given $p, q \in \mathbb{R}$. We managed to solve the L_p dual Minkowski problem for $p > 1$ and $q > 0$ among any convex body, not only the symmetric ones, and obtained results about the regularity of the solution[8].

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[3]K.J. Böröczky, Hai T. Trinh: The planar L_p-Minkowski problem for $0 < p < 1$. Adv. Applied Mathematics, 87 (2017), 58-81.

[4]G. Bianchi, K.J. Böröczky, A. Colesanti, D. Yang: The L_p-Minkowski problem for $-n < p < 1$ according to Chou-Wang. Adv. Math., 341 (2019), 493-535.

[5]G. Bianchi, K.J. Böröczky, A. Colesanti: The Orlicz version of the L_p dual Minkowski problem on S^n for $-n < p < 0$. Adv. Applied Mathematics, 111 (2019), 101937.

[6]K.J. Böröczky, M. Henk, H. Pollehn: Subspace concentration of dual curvature measures of symmetric convex bodies. Journal of Differential Geometry, 109 (2018), 411-429.

[7]K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Yiming Zhao: The dual Minkowski problem for symmetric convex bodies. Adv. Math., 356 (2019), 106805.

[8]K.J. Böröczky, F. Fodor: The L_p dual Minkowski problem for $p > 1$ and $q > 0$. Journal of Differential Equations, 266 (2019), 7980-8033.

2 Miscellaneous results about Stochastic and Analytic Geometry

Valuations

Valuations are finitely additive measures on the space of convex bodies (or on a suitable subspace) valued at an Abelian semigroup. For example, the measures appearing in the dual L_p Minkowski problem are valuations. Since Hadwiger's classical characterization of the intrinsic volumes as the fundamental real valuaed continuous translation and $\mathrm{SO}(n)$ invariant valuations, a deep theory has develoved via the work of McMullen, Alesker and Ludwig.

We managed to characterize $\mathrm{SL}(n,\mathbb{Z})$ equivariant translation invariant Minkowski (convex compact set valued) valuations on lattice polytopes[9]. This result has initiated new research direction about valuations on lattice polytopes parallel to the existing theory develoved by Alesker and Ludwig for valuations on convex bodies.

Based on Alesker's theory on continous valuation on convex bodies and some classical representation theory, we characterized $\mathrm{SL}(m,\mathbb{C})$ equivariant and translation covariant continuous tensor valuations[10].

L_p zonoids and the Brascamp-Lieb inequality

The reverse isoperimetric inequality, due to K.M. Ball, states that if K is an n-dimensional convex body, then there exists an affine image K' of K for which $S(K')^n/V(K')^{n-1}$ is bounded from above by the corresponding expression for a regular n-dimensional simplex, where S and V denote the surface area and volume functional. It was shown by Franck Barthe that the upper bound is attained only if K is a simplex. The discussion of the equality case is based on the equality case in the geometric form of the Brascamp-Lieb inequality. We established stability versions of the reverse isoperimetric inequality and of the corresponding inequality for isotropic measures[11].

We strengthened the volume inequalities for L_p zonoids of even isotropic measures and for their duals, which inequalities are due to Ball, Barthe and Lutwak, Yang, Zhang. Along the way, they proved a stronger version of the Brascamp-Lieb inequality for a family of functions that can approximate arbitrary well some Gaussians when equality holds. The special case $p=\infty$ yields a stability version of the reverse isoperimetric inequality for centrally symmetric bodies[12].

Random objects

We proved asymptotic upper bounds on the variance of the number of ver-

[9] K.J. Böröczky, M. Ludwig: Minkowski valuations on lattice polytopes. JEMS, 21 (2019), 163-197.

[10] J. Abardia, K.J. Böröczky, M. Domokos, D. Kertész: $\mathrm{SL}(m,\mathbb{C})$ equivariant and translation covariant continuous tensor valuations. J. Func. Analysis, 276 (2019), 3325-3362.

[11] K.J. Böröczky, D. Hug: Isotropic measures, and stronger forms of the reverse isoperimetric inequality. Transactions of AMS, 369 (2017), 6987-7019.

[12] K.J. Böröczky, F. Fodor, D. Hug: Strengthened volume inequalities for L_p zonoids of even isotropic measures. Trans. AMS, 371 (2019), 505-548.

tices and missed area of inscribed random disc-polygons in smooth convex discs whose boundary is C^2_+. They also considered a circumscribed variant of this probability model in which the convex disc is approximated by the intersection of random circles[13].

The classical Dvoretzky–Rogers lemma provides a deterministic algorithm by which, from any set of isotropic vectors in Euclideand-space, one can select a subset of vectors whose determinant is not too small. We provided a probabilistic proof of the improvement of Pełczynski and Szarek on Dvoretzky–Rogers lemma, and provide a lower bound for theprobability that the volume of such a random parallelotope is large[14].

Angles

An affirmative answer to the question whether bubbles can be seen in a foam in the title was proved in the plane by showing that any real analytic multicurve can be uniquely determined from its generalized visual angles given at every point of an open ring around the multicurve[15].

We proved that a Hilbert geometry is hyperbolic if and only if the perpendicular bisectors or the altitudes of any trigon form a pencil. We also proved some interesting characterizations of the ellipse[16].

It is well known that the vertices of any Euclidean simplicial regular polytope determine an optimal packing of equal spherical balls. We proved a stability version of optimal order of this result. The method of the prof is analytic using spherical harmonics[17].

Planar geometry

We proved a combinatorial geometric property having its roots in algebraic lattice theory[18], that Ceva's and Menelaus' theorems are valid in a projective-metric space if and only if the space is any of the elliptic geometry, the hyperbolic geometry, or the Minkowski geometries[19], and that Euler's ratio-sum formula is valid in a projective metric space if and only if it is either elliptic, hyperbolic, or Minkowskian[20].

[13] Ferenc Fodor, Viktor Vigh: Variance estimates for random disc-polygons in smooth convex discs, J. Appl. Probab., 55 no 4,, 2018

[14] Fodor, Ferenc; Naszódi, Márton; Zarnócz, Tamás: On the volume bound in the Dvoretzky-Rogers lemma, Pacific J. Math. 301, no. 1, 89–99, 2019

[15] Á. Kurusa: Can you see the bubbles in a foam?, Acta Sci. Math. (Szeged), 82:3-4 (2016), 663-694., 2016

[16] J. Kozma, Á. Kurusa: Hyperbolic is the only Hilbert geometry having circumcenter or orthocenter generally, Beiträge zur Algebra und Geometrie, 57:1 (2016), 243-258., 2016

[17] K. Boroczky, K.J. Boroczky, Alexey Glazyrin, Agnes Kovacs: Stability of the simplex bound for packings by equal spherical caps determined by simplicial regular polytopes, Discrete Geometry and Symmetry. Springer, 2018, 31-60., 2018

[18] Czédli, Gábor; Kurusa, Árpád: A convex combinatorial property of compact sets in the plane and its roots in lattice theory, Categ. Gen. Algebr. Struct. Appl. 11, no. 1, 57–92, 2019

[19] Kurusa, Árpád: Ceva's and Menelaus' theorems in projective-metric spaces, J. Geom. 110, no. 2, Art. 39, 12 pp., 2019

[20] Kurusa, Árpád; Kozma, József: Euler's ratio-sum formula in projective-metric spaces, Beitr. Algebra Geom. 60, no. 2, 379–390, 2019