FINAL PROFESSIONAL REPORT

for the K-115434 OTKA grant proposal

Developing and applying new methods to solving the Cauchy problem in general relativity

1. Introduction

Our research was centered on using new dynamical variables in investigating solutions to the constraint equations and various dynamical systems in general relativity. We have met the project's original objectives and, in many respects, have gone beyond these objectives.

The research proposal was made and approved as a one-person performance. The project had been extended by the joining of my Ph.D. student, Károly Csukás, in 2016. This also enlarged somewhat the scope of the project. The investigations of the evolution of linear perturbations on rotating black hole backgrounds became part of our project. There were two additional unforeseeable alterations in the original plans. First, as a recognition of my recent scientific results, I was awarded a European Scholarship. Besides its good aspects, I was forced to suspend other research grants, and for this reason, the current OTKA grant was suspended for a year. Besides, because of the European Research Organization stipulation, I could not mention the OTKA grant in some of my scientific publications. The other more significant change is rooted in the current worldwide pandemic situation. The close-down periods and the regulations not allowing foreign visits made it impossible to use our research grants as planned. The lack of personal contacts with our foreign colleagues also yielded slight delays in finishing the investigations and writing up our results.

Our main scientific results can be grouped as listed below. They will be outlined in more details in the succeeding subsections.

- the evolutionary form of the constraints
- asymptotically flat solutions to the evolutionary form of the constraints
- black hole initial data: analytic and numerical studies
- the Hamiltonian and canonical gravity aspects
- time evolution: analytic and numerical aspects

2. The evolutionary form of the constraints

I investigated the constraint equations in the case of generic gauge choice. The relevant results are covered by the publication (István Rácz, CQG 2016). In this work, the constraint equations for arbitrary smooth (n + 1) dimensional (with $n \geq 3$) Riemannian or Lorentzian spaces satisfying the Einstein field equations were considered. Regardless of the ambient space signature, I put the constraints into the form of an evolutionary system. It is worth emphasizing that these results were breathtaking even in the Lorentzian case. For the last seven decades, the constraint equations were considered elliptic equations, and they were solved exclusively by applying techniques based on this elliptic character. Therefore, it was a big breakthrough when I put the momentum constraints into a first-order symmetric hyperbolic system for the vector and the trace of the tensorial projections of the extrinsic curvature. The new formalism was completed when I put the Hamiltonian constraint into a parabolic or an algebraic equation for the three-dimensional lapse or the scalar projection of the extrinsic curvature, respectively. What is even more important, I showed that the new form of the coupled Hamiltonian and momentum constraints form either a coupled parabolic-hyperbolic or an algebraic-hyperbolic system. In both cases, I showed the (local) existence and uniqueness of solutions provided a suitable choice for the freely specifiable part of the

data made. Besides, I derived those conditions ensuring the global existence of solutions to the Bernoulli type parabolic equation when the parabolic-hyperbolic form of the constraints is applied (István Rácz, CQG 2016).

3. Asymptotically flat solutions to the evolutionary form of the constraints

As mentioned above, since Lichnerowicz and York's pioneering discovery on the conformal method, the constraint equations were almost exclusively treated as a coupled semi-linear elliptic system. Though five years ago, I introduced two alternative evolutionary methods, so far, only limited analytic investigations had been carried out by others. This section is to summarize our efforts in this direction.

We investigated the hardest analytic problems by attempting to prove the global existence and uniqueness of the solution to these evolutionary equations in the case of near Schwarzschild configurations. Using the techniques of energy estimates in collaboration with Philippe LeFloch, from the University of Sorbonne, and Armando Cabrera, from Tübingen University, we have carried out investigations of the global existence fall off properties of the pertinent solutions. The corresponding results will be published soon (LeFloch P, Rácz I., in preparation), (Cambera A, Rácz I., in preparation).

In parallel to these efforts, together with my Ph.D. student, Károly Csukás, we carried out a systematic numerical investigation of the asymptotic behavior of solutions to the evolutionary form of the constraints (Csukás K, Rácz I. CQG 2020). Note that while solving the evolutionary form of the constraints, besides the freely specifiable variables, we can control only the initial data for the evolutionary equations on one of the leaves of the foliating two-surfaces. Thereby, it was not at all obvious that we can produce asymptotically flat initial data by applying the evolutionary form of the constraints. Whence we started by investigating near Schwarzschild configurations, i.e., we assumed first that each of the freely specifiable variables is the same as they are on t_{KS}=const Kerr-Schild time-slice. However, it turned out that this choice is too restrictive as this does not allow the monopole part of the trace of the tensorial projection of the extrinsic curvature to fall-off as fast as required by asymptotic flatness. We showed that both in the parabolic-hyperbolic and in the algebraic-hyperbolic cases, a slight modification of the freely specifiable part of the data such that asymptotic flatness does not get to be affected by these modifications helps and in (Csukás K, Rácz I., CQG 2020) we produced truly asymptotically flat initial data using the parabolic-hyperbolic and the algebraic-hyperbolic equations.

The corresponding technical achievements are economized now by my Ph.D. student, Károly Csukás, who investigates the more general problem of near Kerr solutions to the constraints equations. Though the technical problems to solve are involved, the results we obtained in the near Schwarzschild case provide useful guidance in investigating near Kerr initial data configurations. The new results of these investigations will also be published with the indication of the present OTKA grant number (Csukás K, in preparation).

All the aforementioned investigations were based on the analytic studies I worked out in collaborations with Jeffrey Winicour. In (Rácz I., Winicour J., arXiv:1601.05386 2016) and (Rácz I., Winicour J., CQG 2018), we reformulated the evolutionary form of the constraints by using a suitable set of spin-weighted variables. In doing so, we used a 2+1 decomposition of the constraints. In this process, quantities intrinsic to the two-dimensional foliating surfaces play a central role. In these works, by assuming that the initial data surface is smoothly foliated either by a one-parameter family of topological two-spheres or topological planes, the basic scalar, vector, and tensorial variables were recast in terms of spin-weighted fields intrinsic to these foliating topological two-surfaces. This, on the one hand, allowes us to

replace all the angular or tangential derivatives in the evolutionary system with some derivatives in the direction of some complex null vector field (these are the well-known Newman-Penrose “eth” and “eth-bar” operators in the spherical case). On the other hand, we can also use an expansion of all the field variables concerning the eigenfunctions of the underlying Laplacian, which, in turn, allows us to rewrite the constraint of the Einstein theory of gravity as a coupled system of ordinary differential equations for expansion coefficients. We applied the corresponding analytic forms of the aforementioned investigation constraints with my Ph.D. student (Csukás K, Rácz I., CQG 2020).

4. Black hole initial data: analytic and numerical studies

We constructed various types of black hole initial data configurations by using the evolutionary form of the constraints. These involved initial data constructions for perturbed individual black holes, but the main focus was on initial data for binary black hole configurations.

We provided comprehensive numerical investigations of black hole initial data by applying the parabolic-hyperbolic form of the constraints with Anna Nakonieczna and Łukasz Nakonieczny (Nakonieczna A., Nakonieczny Ł., Rácz I., arXiv:1712.00607, 2017, submitted to CQG). These investigations of individual spinning and boosted black holes were based on the mathematical model I worked out for more general configurations (Rácz I., Astronomy Reports, 2018).

In (Rácz I., Astronomy Reports, 2018), I provided a detailed analytic setup suitable to describe physically realistic binary black hole initial data with aligned or anti-aligned spins by applying the parabolic-hyperbolic formulation of the constraints and by superposing Kerr-Schild black holes. The method is surprisingly simple and has the advantage that its input parameters are essentially the same as those used in the post-Newtonian (PN) setup, thereby providing various interrelations between these two important but distinct formalisms. My construction is also distinguished by that, as opposed to any of the other constructions, no “ad hoc” boundary conditions in the strong-field regime are applied.

Besides, in (Rácz I., arXiv:1608.02283, 2016), I proved that the PN-like input parameters determine all the global ADM charges. Notably, the input parameters' specific choice allows us to determine the ADM quantities of the to-be solutions explicitly in terms of the separations velocities and spins of the individual Kerr black holes in advance of solving the constraints.

5. The Hamiltonian and canonical gravity aspects

The new results in (Rácz I, arXiv:1912.02576, 2019) have important impacts on time's classical and quantum nature. To understand the significance of these results, one should recall that the energy and time, in general, are of distinguished importance in the canonical setup. On the one hand, in geometric theories of gravity such as Einstein’s theory, not only the external time parameter (think about the concept of absolute time in Newtonian physics) is missing, but we do not even have appropriate concepts of the energy that could allow us to measure, e.g., the energy of bounded three-dimensional regions. On the other hand, time and energy are canonically conjugate quantities, whence they are fundamental concepts in canonical gravity. Indeed, the geometric construction in (Rácz I, arXiv:1912.02576, 2019) is related to this central problem of canonical gravity.

The new results we achieved in collaboration with Dr. Przemysław Małkiewicz from the National Centre for Nuclear Research, are also remarkable as they make progress in finding a proper implementation of harmonic gauge in canonical gravity (Przemysław M., Rácz I., in preparation). The spacetime harmonic gauge condition is usually given via spacetime

coordinates. In (Przemysław M., Rácz I., in preparation), we give a proper representation of the canonical approach's harmonic gauge condition. More precisely, we gave the corresponding consistency relations by introducing suitable evolution equations for the lapse and the shift.

In (Bengtsson I., Rácz I., in preparation), in collaboration with Professor Ingemar Bengtsson from Stockholm University, we put the tetrad-based constraints into evolutionary form. To understand the significance of this result, recall that in Dirac's proposal to quantize constrained systems, the constraints are ignored at the classical level. The physical quantum states are supposed to be determined by solving the quantum constraints on a kinematical Hilbert space. In principle, this method could be applied to the constraints in Einstein's theory of gravity. As a radically new alternative to this approach, we decided to solve the classical level constraints and construct the phase space associated with the gravitational degrees of freedom. In the end, it will be endowed with a natural cotangent-bundle structure. The particular model we worked with is canonical gravity based on the use of Ashtekar-Barbero-Thiemann variables which starts by working with an orthonormal triad. We showed that the constraints can always be solved as an evolutionary system even using these variables. This result is expected to significantly boost the loop quantum gravity developments, where very few real breakthroughs have occurred recently.

6. Time evolution: analytic and numerical aspects

In (Csukás K., Tóth G.Zs., Rácz I., PRD, 2020) and (Csukás K., Rácz I., accepted in PRD, 2021), we investigated the evolution of spin s = 0, ±1, ±2 fields by solving the homogeneous Teukolsky equation numerically. We investigated the late time tail behavior, as well as superradiance. Due to the applied conformal compactification, the asymptotic decay rates, along with energy and angular momentum fluxes, could be determined at three characteristic locations—in the domain of outer communication, at the event horizon, and at future null infinity—simultaneously.

Our investigations are pioneering in the following respects. We carried out the first fully generic (without using any symmetry reduction) investigation of the evolution of spin s = 0, ±1, ±2 perturbations of rotating black hole background. Second, in (Csukás K., Tóth G.Zs., Rácz I., PRD, 2020) and (Csukás K., Rácz I., accepted in PRD, 2021), we applied a new type of 'energy' and 'angular momentum' balance relations,—that can be associated with any pair of spin 's' and spin '-s' solutions to Teukolsky equation—to demonstrate the feasibility of the numerical schema, and also to verify the proper implementation of the underlying mathematical model.

In (Csukás K., Tóth G.Zs., Rácz I., PRD, 2020) and (Csukás K., Rácz I., accepted in PRD, 2021) the applied numerical setup was developed in collaboration with my PhD student Károly Csukás. We used a combination of conformal compactification and the hyperbolic initial value problem. The framework is generic as it applies to spacetimes that are smoothly foliated by a two-parameter family of topological two-spheres—in the Kerr case, these are the Boyer-Lindquist 't=const' and 'r=const' ellipsoids—such that the evolved fields are expanded in terms of spin-weighted spherical harmonics. This allowed us to evaluate all the angular derivatives analytically as they can be expressed either in terms of the `eth' and `ethbar' operators and derivative with respect to the Kerr background's axial Killing vector field. The multiplication of the functions is determined using the Gaunt coefficients, while the division by non-vanishing spin-zero quantities is traced back to multiplication using Neumann-series. The expansion coefficients were then evolved, in the time-radial section, by applying the method of lines. We showed that the corresponding finite difference part of the code is of fourth-order accurate (Csukás K., Tóth G.Zs., Rácz I., PRD, 2020), while we showed that

that in the angular directions, the convergence rate, as expected, is exponential (Csukás K., Tóth G.Zs., Rácz I., PRD, 2020) and (Csukás K., Tóth G.Zs., Rácz I., accepted in PRD 2021).

The documentation of the program packages is available at the following links:

ssh: //git@gitlab.kfki.hu: 2222 / csukas.karoly / constraints.git

ssh: //git@gitlab.kfki.hu: 2222 / csukas.karoly / Teukolsky.git

5. Summary

In our research plan, we made commitments for publishing about 4-6 research papers as a total in high-quality scientific journals. This has been fulfilled more than twice by publishing 15 papers of this type. Among these, 9 have already been published as Q1-rated, 1 as Q2-rated journal articles, and 5 pending papers are to be published in similarly rated journals. I finished 5 of these 15 research papers while I had to suspend the OTKA grant for a year. Because of the European Research Organization stipulation, I could not mention the OTKA grant in these 5 scientific publications.

A significant achievement of our research project is that my doctoral scholarship, Károly Csukás, who joined the project after the start of the OTKA-supported research program, will soon complete his doctoral training. He fulfilled all the conditions, took the necessary language exams, passed his exams in the professional subjects with a summa cum laude qualification. He prepared his doctoral dissertation that the referees currently examine. Presumably, even before June this year, he will be awarded the Ph.D. degree by the Doctoral Committee of Eötvös Loránd University.

Our results were presented in 21 invited conference lectures and 19 invited institutional seminar presentations besides our scientific publications. I was also invited to give university courses at two internationally outstanding foreign universities. The participants of these were Ph.D. and postdoctoral students of those universities. The list of publications has been uploaded to the online system, while our conference, seminar, and invited course presentations are listed below.

Conference lectures and invited talks

1. Rácz I.: Constraints as evolutionary systems,
   21. International Conference on General Relativity and Gravitation; New York, 13 July, 2016
2. Rácz I.: A new method of constructing binary black hole initial data
   Workshop on Numerical and Mathematical Relativity, Oppurg, Germany, 8-10 September 2016
3. Rácz, I.: Constraints as evolutionary systems
   Conference on Recent Developments in Gravity, Mykonos, Greece 19 September, 2016
4. Rácz, I.: The many faces of the constraints
   3rd POTOR Conference, Krakow, 28 September, 2016
5. Rácz I.: On the use of evolutionary methods in spaces of Euclidean signature
   15th Spanish-Portuguese Relativity Meeting, Málaga, Spain, 14 September 2017
6. Rácz I.: A new method of constructing binary black hole initial data
   PoToR 4 Conference, Kazimierz Dolny, Poland, 2017 September 27
7. Rácz I.: Constraints as evolutionary systems
   Chinese Academy of Sciences, Morning Site Center, Beijing, 6 November 2017
8. Rácz I.: A new method of constructing binary black hole initial data
   Chinese Academy of Sciences, Morning Site Center, Beijing, China, 6 November 2017
9. Csukás K.: Notes on geometric inequalities in spherically symmetric spacetimes
   Spanish-Portuguese Relativity Meeting, Málaga, Spain, 12-15 September 2017
10. Csukás K.: Some results on geometric inequalities in spherically symmetric spacetimes
    8th Central European Relativity Seminar, Brno, Czech Republic, 15-17 February 2018
11. Rácz, I.: Evolution in spaces of Euclidean signature?

Lepold Infeld Colloquium, FUW, University of Warsaw, 22 March, 2018

12. Rácz, I.: A new method of constructing binary black hole initial data
    The Third Zeldovich Meeting, April 23-27, 2018, Minsk, Belarus, 24 April, 2018
13. Rácz, I.: On the nature of spacetime singularities
    The 2nd workshop on "Singularities of general relativity and their quantum fate"
    Banach Center, 21.05.2018 - 25.05.2018 Warsaw, 22 May, 2018
14. Rácz, I.: Construction of Riemannian three spaces with quasi-convex foliations and with non-decreasing Geroch mass
    Marian Smoluchowski Institute of Physics, Jagellonian University, Kraków, 17 October, 2018
15. Rácz, I.: A new method of constructing binary black hole initial data
    Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada, 6 March, 2019
16. Rácz, I.: On the use of evolutionary methods in spaces of Euclidean signature
    Grav19, UNC, Córdoba, Argentina, 2019.04.08--2019.04.12, 11 April, 2019
17. Rácz, I.: Symmetries of vacuum spacetimes with a compact Cauchy horizon of constant non-zero surface gravity
    22nd International Conference on General Relativity and Gravitation, Valencia, Spain, 9 July, 2019
18. Csukás K.: On the asymptotics of initial data by evolutionary solvers
    22nd International Conference on General Relativity and Gravitation, Valencia, Spain, 7-12 July 2019
19. Rácz, I.: On the use of evolutionary methods in spaces of Euclidean signature
    Chinese Academy of Mathematics and System Sciences, MCM, Beijing, China, 25 October, 2019
20. Rácz, I.: On the construction of Riemannian three-spaces with smooth generalized inverse mean curvature flows
    General Relativity, Geometry and Analysis: beyond the first 100 years after Einstein, Institut Mittag-Leffler, Stockholm, Sweden 5 December 2019
21. Rácz, I.: The many faces of the constraints
    Department of Mathematics and Mathematical Statistics, Umeå University, Sweden 10 December, 2019

Invited institutional seminars

1. Rácz I.: The many faces of the constraints in general relativity
   Institute of Physics, Maria Curie-Sklodowska University, Lublin, Poland, 27 October, 2015
2. Rácz I.: The many faces of the constraints in general relativity
   Institute of Theoretical Physics, Faculty of Mathematics and Physics Charles University, Prague, Czech Republic, December 1, 2015
3. Rácz I.: The many faces of the constraints in general relativity
   Gravitational Physics Department Faculty of Physics, University of Vienna, December 3, 2015
4. Rácz I.: Constraints as evolutionary systems
   Albert Einstein Institute, Golm, Germany, 6 April, 2016
5. Rácz I.: A simple method of constructing binary black hole initial data
   Albert Einstein Institute, Golm, Germany, 27 April, 2016
6. Rácz I.: Constraints as evolutionary systems
   Theoretical Physics Department, Seminar of Research Training Group, University of Jena, Germany, 7 June, 2016
7. Rácz I.: A new method of constructing binary black hole initial data
   Faculty of Physics, University of Warsaw, Warsaw, Poland, 18 November, 2016
8. Rácz, I.: On the use of evolutionary methods in spaces of Euclidean signature
   Seminar of Theory of Relativity and Gravitation, FUW, University of Warsaw, Poland, 23 March, 2018
9. Rácz, I. Date:
   Title: Stationary black holes as holographs
   Seminar of Theory of Relativity and Gravitation, FUW, University of Warsaw, Poland, 18 May, 2018
10. Rácz, I.: A new proof of the positive mass theorem
    IX International Meeting on Lorentzian Geometry, Banach Center, 17.06.2018 - 24.06.2018. Warsaw, Poland, 20 June, 2018
11. Rácz, I.: Construction of quasi-convex foliations with monotonous Geroch mass
    The 5th Conference of the Polish Society on Relativity, 24-28.09.2018, Wojanów, 24 September, 2018
12. Rácz, I.: On the use of evolutionary methods in spaces of Euclidean signature
    Relativity Group at the University of Chicago, Chicago, USA, 26 February, 2019
13. Rácz, I.: Fully constraint time evolution in Einstein theory
    Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada, 5 March, 2019
14. Rácz, I.: Construction of initial data with monotonous Geroch mass
    Geometrische Analysis, Differentialgeomtrie und Relativitätstheorie Fachbereich Mathematik Universität Tübingen,Germany, 7 February, 2019

15. Rácz, I.: Construction of initial data with monotonous Geroch mass
    Seminar of Theory of Relativity and Gravitation, University of Warsaw, Poland, 17 May, 2019
16. Rácz, I.: On the use of evolutionary methods in spaces of Euclidean signature
    Physics Department, Bogazici University, Istambul, Turkey, 1 July, 2019
17. Rácz, I.: Construction of Riemannian three-geometries with monotonous Geroch mass
    Department of Mathematics, Fudan University, Fudan, China, 21 October 2019
18. Csukás K.: Time evolution in all directions
    Simonyi Day, Wigner Research Center for Physics , Budapest, Hungary, 19 October 2020
19. Rácz I.: The many faces of the constraints in GR
    Wigner Research Center for Physics, Budapest, Hungary, 16 December 2020

Invited courses at universities:

1. Rácz I.: Lecturing at Tsinghua University, Beijing, China
   An intensive courses for doctoral and postdoc students
   2017/10 - 2017/11 On the use of evolutionary methods in spaces of Euclidean signature
2. Rácz I.: Lecturing at the University of Warsaw, Warsaw, Poland
   Monographic lectures for PhD students and postdoctoral fellows
   2018/09 - 2019/02 On the use of evolutionary methods in metric theories of gravity,
   Course ID: 1102-4EMMTG Erasmus code / ISCED: 13.204/ (0533) Physics

Poszter:

1. Csukás K.: On the asymptotics of solutions to the evolutionary form of the constraints,
   9th Central European Relativity Seminar,
   Krakow, Poland, 14-16 February 2019