K115383 – Project closing report

The problems we were working on are connected to several branches of mathematics, mainly to functional analysis, but also to algebra and geometry. The essential part of the work was devoted to the study of preserver transformations on structures of operators, on quantum structures, and on metric spaces of measures. The results obtained in the course of the project were published in 75 papers. As it can be seen from the detailed list of references, a substantial part of them were published in highly respected journals (with Scimago rank Q1) among which are several top journals (with Scimago rank D1) like Advances in Mathematics, American Journal of Mathematics, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, International Mathematics Research Notices, Journal of Functional Analysis, SIAM Journal on Mathematical Analysis, and the Transactions of the American Mathematical Society.

In order to give a systematic overview, we group our results as follows:

I. Isometries and generalized isometries [1,2,15,16,20,22,29,30,32,33,35,42,43,44,58,67,68,70]
II. Isomorphisms and local maps [5,10,47,52,60,61,63,72]
III. Results concerning means [12,21,24,34,36,38,39,45,49,51,62,64,65,71,73]
IV. Preserver problems on quantum structures [4,7,8,9,14,19,23,25,26,27,37,46,56,57,69]
V. Results concerning mathematical physics [50,53]
VI. Further miscellaneous works [3,6,11,13,17,18,28,31,40,41,48,54,55,59,66,74,75]

In what follows, we briefly summarize the obtained results. Actually, due to the strict space limitations, we can highlight only a fraction of them.

I. Isometries and generalized isometries

Some of our works concern maps preserving certain important numerical quantities that appear in the theory of quantum information. Those investigations were basically motivated by Wigner's celebrated theorem on quantum mechanical symmetry transformations. That result, which is particularly important in the probabilistic aspects of quantum theory, asserts that every bijective transformation of the set of all rank-one projections on a (complex, not necessarily finite dimensional) Hilbert space which preserves the transition probability (the trace of the product which can also be interpreted as the squared cosine of the angle between the ranges of those projections) is implemented by either a unitary or an antiunitary operator on the underlying Hilbert space. This famous result has recently been generalized in many different ways. For example, in 2013 Botelho, Jamison and Molnár established a characterization of surjective isometries of Grassmann spaces of all projections of a fixed rank which leave the so-called gap distance invariant. That quantity between rank-one projections is a simple function of the trace of the product of the projections. Hence, the above mentioned result can really be viewed as a generalization of Wigner's original theorem. However, in the paper by Botelho, Jamison and Molnár there were some mild dimensional restrictions. In [1] we were able to remove those restrictions and gave a characterization of the surjective isometries of Grassmannians with respect to the gap metric. The method given in [1] is more elementary than the one used in the paper by Botelho, Jamison and Molnár, and it is applicable for underlying spaces of any dimension. The new argument works in the real case as well, and in finite dimension we were able to prove our characterization even without assuming the surjectivity of the transformation. In the paper [30], we extended these results for the case of Grassmannians of all projections with infinite rank and infinite corank. Those papers appeared in the prestigious journals Journal of Functional Analysis and Advances in

Mathematics.

We also highlight [22], where Gaál and Guralnick (recent awardee of the Frank Nelson Cole prize of AMS) described the structure of isometries of the linear space of self-adjoint traceless matrices. Their approach was based on representation theoretic methods and was applicable to determine the isometries of the space of skew-symmetirc matrices as well. In [67] the same authors proved that when a compact simple Lie group G acts absolutely irreducibly on a vector space V, almost always the connected stabilizer of a G-invariant norm coincides with either G or the whole $\mathrm{SO}(V)$, and the short list of exceptional cases was determined.

In [33] we described the Bures-Wasserstein isometries of the positive definite cones and also the so-called density spaces in C^*-algebras. The Bures-Wasserstein metric is important for its applications in quantum information theory, optimization, and optimal transport. We proved that those isometries are restrictions of Jordan *-isomorphisms between the underlying C^*-algebras multiplied by certain central elements.

Another direction in our research concerning isometries deals with (not necessarily bijective) distance preserving maps on various metric spaces of probability measures. In [29] we described the structure of those bijections of the space of probability measures on the real line that preserve the Kuiper distance. This work was motivated by an earlier result of Molnár, in which he described the structure of isometries with respect to the Lévy metric. In [43] we conducted similar investigations. We considered the collection of all Borel probability measures on a separable real Banach space. The aim was to determine the form of the surjective isometries on that set corresponding to the Lévy-Prokhorov metric. This distance is important because it metrizes the topology of weak convergence of continuous stochastic processes. It turned out that those isometries are intimately connected to the surjective isometries of the underlying Banach space. That paper appeared in Ann. Scuola Norm. Sup. Pisa Cl. Sci.

Another, nowadays very important and closely studied metric is the Wasserstein metric. A recent result of Kloeckner states that the isometry group of the Wasserstein space $W_2(\mathbb{R})$ is exceptional among all $W_2(\mathbb{R}^n)$ spaces. Namely, $W_2(\mathbb{R})$ admits an exotic isometry flow. We showed in the paper [58] (appeared in Trans. Amer. Math. Soc.) that the space $W_2(\mathbb{R})$ is exceptional in regard of the parameter $p = 2$, too. Namely, the space $W_p(\mathbb{R})$ admits exotic isometries if and only if $p = 2$. The main goal in [48] was to compute the isometry groups of the Wasserstein spaces $W_p(\mathbb{R})$ and $W_p([0, 1])$ for all values of the parameter p. In [70], a broad generalization of this result was given. Namely, we described the isometry group of those p-Wasserstein spaces that are built on (possibly infinite dimensional) separable real Hilbert spaces. Furthermore, we constructed a Wasserstein space that possesses ill behaving isometries. Such isometries are not induced by a single map in the push-forward sense, and they do not map the set of Dirac masses into themselves (in other words, there are isometries that split mass). The existence of such spaces and isometries was an open problem posed by Kloeckner.

Finally, we highlight one of our four papers [2,15,16,20] about generalized isometries. Namely, [15] is concerned with certain maps on positive definite cones of matrices. The generalized distance measures (GDM-s for short) what those maps preserve are kinds of distances connected to quantum divergences, relative entropies, or related to geometric structures on that cone. Quite uniquely, we did not suppose there that the generalized isometries under considerations are surjective. However, in some cases those maps turned out to be automatically onto while in other cases we could determine the exact structure of all not necessarily surjective transformations leaving a given GDM invariant.

II. Isomorphisms and local maps

In this group we collect those results that concern various types of isomorphisms and local maps. In [5] we determined the structure of all continuous endomorphisms on the cone of 2×2 complex positive definite matrices with respect to the operation of the Jordan triple product. In the case of matrices of larger size, the form of those transformations was determined some time ago. Surprisingly, the problem remained open for $n = 2$ although several attempts were made to solve it. The importance of the 2×2 case lied in that we wanted to complete some of our earlier works on generalized isometries in that low dimensional case. In [5] we managed to give a precise description of those maps. Strangely enough, the proof in the 2-dimensional case was significantly more difficult than in the higher dimensional cases.

In [61] we presented characterizations of isomorphisms of Jordan algebras of quantum observables using only their certain spectrum-preserving properties without assuming any kind of linearity. In particular, we proved that a bijective map between the self-adjoint parts of von Neumann algebras preserves the spectrum of products if and only if it equals a (linear) Jordan-isomorphism multiplied by a fixed central self-adjoint unitary element. In the first part of the survey paper [63] certain preserver properties of Jordan *-isomorphisms on C^*-algebras are discussed and classical results are presented showing that, to some extents, those properties in fact characterize Jordan *-isomorphisms among bijective linear transformations. In the second part some recent results are given which can be regarded as non-linear characterizations of Jordan *-isomorphisms on positive cones in operator algebras or on the spaces of self-adjoint elements.

Finally, we highlight [72], the latest one of our papers [47,60,72] concerning local maps. In a very general setting, we introduced a new type of local maps there, a new sort of reflexive closure of a given class of transformations relative to a given operation that we call operational reflexive closure, and a corresponding concept of reflexivity. We calculated the operational reflexive closures of some important classes of transformations and significantly strengthen former 2-reflexivity results concerning the automorphism groups of various operator structures. A typical result among them is the following: if φ is a map from the unitary group over a separable infinite dimensional Hilbert space into itself with the property that for any pair V, W of unitaries there is a group automorphism $\alpha_{V,W}$ of the unitary group such that $\varphi(V)\varphi(W) = \alpha_{V,W}(VW)$, then either φ itself or $-\varphi$ is a group automorphism. This result apparently and substantially generalizes a former one on the 2-reflexivity of the automorphism group of the unitary group.

III. Results concerning means

During the course of the project we have obtained various types of results concerning means: characterizations of means, characterizations of maps that preserve means, and descriptions of maps that preserve some quantities related to means. In what follows, we highlight the most important ones among the corresponding publications, see above.

In [45] we presented some characterizations for quasi-arithmetic operator means (among them the arithmetic and harmonic means) on the positive definite cone of the full algebra of Hilbert space operators, and also for the Kubo-Ando geometric mean on the positive definite cone of a general non-commutative C^*-algebra. Among others, we proved that on the positive definite cone of a full Hilbert space operator algebra, the only Kubo-Ando means that can be transformed to the arithmetic mean are the arithmetic and the harmonic means. We also obtained that on the positive definite cone of a non-commutative C^*-algebra, exactly the geometric mean has the following property: the process similar to the one along which we get quasi-arithmetic means from the arithmetic mean, produces no new Kubo-Ando mean.

The main results of [12] concern maps between the positive cones of C^*-algebras transforming the arithmetic, geometric and harmonic means into themselves or into each other. We presented the structure of the automorphisms of positive cones with respect to the arithmetic and harmonic means in the case of C^*-algebras, and we described the contiuous automorphisms with respect to the geometric mean in the case of von Neumann factors. In the first two cases, the conclusion was that the maps are closely related to the Jordan *-automorphisms of the underlying algebras while in the third case the conclusion was more complicated, also a certain abstract notion of the determinant showed up. We also obtained the structure of all continuous maps from a von Neumann factor to a Banach space which transform the geometric mean to the arithmetic mean. As a byproduct, that result also gave us an interesting common characterization of finite von Neumann factors, the trace functional, and the classical logarithmic function.

In the paper [71] we studied transformations between positive definite cones in operator algebras preserving the spectral geometric mean of Fiedler and Pták. Under some regularity conditions, we proved that any bijective such map on the positive definite cone of the full operator algebra over a Hilbert space is necessarily a unitary or antiunitary similarity transformation. We also described the continuous functionals with respect to that mean as an operation on the positive definite cones in von Neumann algebras as well as the surjective maps on the positive definite cones in general C^*-algebras preserving the norm of that mean. We also discussed kinds of structural similarities and dissimilarities between the Kubo-Ando geometric mean and the spectral geometric mean.

The primary aim of [73] was to present the complete description of the isomorphisms between positive definite cones of C^*-algebras with respect to the recently introduced Wasserstein mean and to show the nonexistence of nonconstant such morphisms into the positive reals in the case of von Neumann algebras without type I_2, I_1 direct summands.

IV. Preserver problems on quantum structures

First we highlight the papers [9],[56] and [69]. Our investigations there were basically motivated by Wigner's celebrated theorem on quantum mechanical symmetry transformations that we have already mentioned above. This famous result has been generalized in various ways by a number of researchers. Among others, in 2001, Molnár provided a natural generalization, namely, he established a characterization of (not necessarily bijective) maps of the Grassmann space of all rank-n projections which leave the system of so-called Jordan principal angles invariant. In [9] we gave a very natural common generalization of Wigner's and Molnár's theorems, namely, we presented a characterization of those (not necessarily bijective) transformations of a Grassmann space which leave the quantity $\mathrm{Tr}PQ$ (i.e., the sum of the squares of cosines of principal angles) invariant for every pair of rank-n projections P and Q. In [56] we investigated transformations on the projective space $P(H)$ and on the unit sphere S_H of real Hilbert spaces H of dimension at least 3, and on the projective space $P(H)$ over complex Hilbert spaces H of dimension at least 2 preserving a given angle. Note that the usual angles define metrics on these spaces. The main result of the paper is that, under some conditions on the fixed angle, these transformations are implemented by unitary or antiunitary operators on H. We point out that in Wigner's fundamental theorem all angles are assumed to be preserved while in our results merely one. Still, the conclusions are the same, except for one of the low dimensional cases. In [69] we solved a problem that remained open in [56]. Namely, we showed that every bijective self-map of a complex projective Hilbert space that preserves the transition probability $0 < p < 1/2$, automatically preserves every transition probability. By Wigner's theorem this implies that such maps are always induced by unitary or antiunitary operators. The latter two papers appeared in Int. Math. Res. Not.

The second subgroup [19,14,46] of papers deals with maps that preserve entropies (we investigated both the classical and C^*-algebra contexts). Further results concerning entropy preservers can be found in [4,7,8,14,19,23,46]. In [19] we determined the form of those bijections on the sets of positive semidefinite/definite and density matrices which preserve the so-called quantum χ^2_α-divergence. In all those cases we proved that the corresponding invariance transformations originate from the algebra *-automorphisms or *-antiautomorphisms of the underlying matrix algebra.

In [14] we investigated transformations on the positive definite cone A^{++} in the C^*-algebra A with a faithful trace which leave a so-called quasi-entropy (a very general notion of quantum divergence) invariant. That quantity is parametrized by a continuous real function f on the positive half-line and by an element w of the algebra A. From the physical point of view the most important quasi-entropies are the ones associated with any of the functions $f(x) = xlogx$, $-logx$ or x^p (p is a given real number). In [14] we gave the precise structure of those bijections on A^{++} which preserve the quasi-entropy associated with one of the latter functions and with an arbitrary invertible element w of A. It turned out that those maps are closely related to the Jordan *-automorphisms of the underlying algebra. In the paper [46] (appeared in J. Funct. Anal.) the definitions of different types of quantum Rényi relative entropy were extended from the finite dimensional setting of density matrices to density spaces of C^*-algebras. The symmetry groups corresponding to each of those numerical quantities were determined. Furthermore, it was shown that those quantities are essentially different on non-commutative algebras, none of them can be transformed to another one by any surjective transformation between density spaces. Similar results concerning the Umegaki and the Belavkin-Staszewksi relative entropies were also presented.

Additional five papers which deal with various preserver problems on quantum structures are [25,26,27,37,57]. Here we highlight only [57]. The probably most fundamental mathematical structure in the theory of unsharp quantum measurements is the so-called effect algebra. In [57] we considered maps on that structure which respect the relation of coexistence (that expresses the joint measurability of unsharp quantum events) and characterized them completely. It turned out that these sort of automorphisms can be quite different from the standard automorphisms of the effect algebra.

V. Results concerning mathematical physics

Beside the mathematical results concerning quantum structures, we have two research papers which belong to the field of mathematical physics.

In [50] we calculated the entanglement entropy of a non-contiguous subsystem of a chain of free fermions. This problem leads to the asymptotic analysis of a complex contour-integral which involves an inverse Toeplitz matrix whose symbol possesses Fisher-Hartwig singularities. The analysis was carried out utilizing the Riemann-Hilbert method.

In the paper [53] (that appeared in SIAM J. Math. Anal.) we investigated discrete logarithmic energy problems in the unit circle. We studied the equilibrium configuration of n electrons and $n-1$ pairs of external protons of charge $+1/2$. It was shown that all the critical points of the discrete logarithmic energy are global minima, and they are the solutions of certain equations involving Blaschke products. As a nontrivial application, we refined a recent result of Simanek, namely, we proved that any configuration of n electrons in the unit circle is in stable equilibrium (that is, they are not just critical points but are of minimal energy) with respect to an external field generated by $n-1$ pairs of protons.

VI. Further miscellaneous works

In this section we collect all those papers which do not belong to any of the previous sections.

We obtained several several results concerning determinant preserving maps [3,17,28,41]. We highlight [28], in which we verified a generalization of the well-known Minkowski's determinant inequality in the context of finite von Neumann algebras. We characterized the case of equality and applied it for the solution of a nonlinear preserver problem relating to the determinant. Namely, we described those bijective maps between the positive definite cones in finite von Neumann algebras which preserve the determinant of sums of elements and proved that those maps are closely related to Jordan *-isomorphisms between the underlying algebras.

In the paper [18] (appeared in Amer. J. Math.), we generalized the theory of gradient flows of semi-convex functions on CAT(0)-spaces, developed by Mayer and Ambrosio-Gigli-Savaré, to the case of CAT(1)-spaces. The key tool here is the so-called "commutativity" representing a Riemannian nature of the space. Our results hold true also for metric spaces satisfying the commutativity with semi-convex squared distance functions. The main applications of the generalized theory include the convergence of the discrete variational scheme to a unique gradient curve, the contraction property and the evolution variational inequality of the gradient flow, moreover a Trotter-Kato product formula for pairs of semi-convex functions.

In [31], we established a Jensen inequality for lower semicontinuous convex functions from a Hadamard space into an ordered Hadamard space. That inequality was applied to matrix valued integrable functions on probability measure spaces. New results on matrix analysis and eigenvalue analysis were provided by discovering a variety of Lipschitz convex functions related to eigenvalue maps and the Löwner ordering of positive definite matrices.

In [13] we characterized those self-adjoint elements x of a C^*-algebra A which are "points of operator monotonicity" of the exponential function. By this we mean the property that if y is a self-adjoint element of A such that $x \leq y$, then $\exp(x) \leq \exp(y)$, where $\leq$ is the usual order on the set of self-adjoint elements in A. Our characterization says, among others, that the points x having this property are exactly the central elements of A. In the paper [48] this result was substantially extended. We proved that for any strictly convex increasing function f defined on an open interval which is unbounded from above, a self-adjoint element x of C^*-algebra A belongs to the center if and only if f is locally monotone at x, i.e., for any self-adjoint element $y \in A$, the relation $x \leq y$ implies $f(x) \leq f(y)$.

In the paper [55] (that appeared in Int. Math. Res. Not.) we gave an answer to David Blecher's problem concerning the characterization of all real operator monotone functions (that are numerical functions which preserve the order for the Hermitian parts of operators). It turned out that the real parts of those functions are operator monotone and if we also require the function to be free holomorphic, then it must be affine with an arbitrary constant part and a completely positive linear part.

In [74] we considered some faithful representations of positive Hilbert space operators on structures of nonnegative real functions defined on the unit sphere of the underlying Hilbert space. Those representations turn order relations between positive operators to order relations between real functions. Two of them turn the usual Löwner order between operators to the pointwise order between functions, another two turn the spectral order between operators to the same, pointwise order between functions. We investigated which algebraic operations those representations preserve, hence which kind of algebraic structure the representing functions have. Most notably, we proved that the representation by strength functions due to Busch and Gudder is a monomorphism under the operation of the Kubo-Ando harmonic mean (or parallel sum). Also, we introduced a new complete metric (which corresponds naturally to the spectral order) on the set of all invertible positive operators and formulated a conjecture concerning the corresponding isometry group.

We conclude the report with the following remarks. First, in the original research proposal we promised ca. 25-30 publications in the course of the project which, as can be seen above, has been far exceeded. Second, beside research papers, there were other kinds of outputs. We reported on the obtained results at high level international conferences and research seminars at prominent universities and institutions like, among others, Chern Institute of Mathematics at Nankai University, ISI Delhi, IST Austria, Jagiellonian University, KU Leuven, Lomonosov Moscow State University, Lorentz Center Leiden, Queen's University Belfast, RIMS Kyoto, Shanghai University, University of Oxford, Yau Mathematical Sciences Center of Tsinghua University, UW Madison. Moreover, we not only participated at conferences but were also involved in the organization of conferences. In 2017, the international conference 'Preservers Everywhere' was organized by the PI and Gy. Gehér in Szeged. The list of keynote speakers included a number of eminent researchers of the field of preservers from Canada, China, Croatia, Czech Republic, Japan, Slovenia, Spain, Taiwan, and the USA. Following a former tradition, an international one-weekend workshop on preserver problems was organized by the PI and Titkos in 2019 in Szeged. Finally, as for the recent past, the PI was an organizer of a minisymposium on preservers at the 8th European Congress of Mathematics in 2021.