Final report

In the framework of this Slovenian-Hungarian joint project, we considered problems about graphs (also hypergraphs), groups and geometries (both finite and classical). In our original research plan we illustrated the problems to be studied by several concrete examples. Some of these problems were solved (as seen from this report), some of them were replaced by related problems. The main direction in graph theory were problems in which connections with geometry played an important role. We studied several coloring problems for (hyper)graphs. Some of the problems were obtained in actual collaboration with Slovenian colleagues, some work was done independently. We note that also the Slovenian colleagues worked independenlty on some of the problems mentioned in our original research plans (mostly the problems related to Cayley graphs or more generally, on group actions on graphs, arc-transitive bicirculants and applications of Schur rings). They published several dozens of papers supported by the Slovenian part of our bilateral project. In this final report, we briefly describe our research topics and review our results accordingly, and then give a brief summary of our other activities carried out in the framework of the project. Let us remark that, compared to the 32 publications listed in the other table, the present specification contains seven additional unpublished manuscripts written in the framework of the proposal, which have not been uploaded to arxiv.org yet.

Colorings.

In the centre of our proposals we had graph colouring problems related to geometry. In particular, we studied the upper chromatic number of the hypergraph coming from the points and the d-dimensional subspaces of $\mathrm{PG}(n,q)$, and also its balanced upper chromatic number. Other versions of the chromatic number (e.g., the chromatic, achromatic and pseudochromatic indices) applied for affine and projective spaces were also considered in our work. The papers related to these questions are the following.

∘ Z.L. Blázsik, T. Héger, T. Szőnyi: On the upper chromatic number and multiple blocking sets of $\mathrm{PG}(n,q)$. *Journal of Combinatorial Designs*, 2019;1–23. `https://doi.org/10.1002/jcd.21686`

We managed to generalise the results of Bacsó, Héger and Szőnyi [J. Comb. Des. 2013] regarding the upper chromatic number of projective planes. We investigated similar questions for those hypergraphs whose point set is the points of the n-dimensional projective space and the hyperedges correspond to k-dimensional subspaces for a fixed k ($n \geq 2$, $1 \leq k \leq n-1$), which hypergraph we denoted by $\mathcal{H}(n,k,q)$. Here one would like to color the points of $\mathrm{PG}(n,q)$ with the most number of colors so that there are no rainbow hyperedges, i.e. every k-dimensional subspace contains at least two points with the same color. We call such colorings proper, and a proper coloring is said to be *trivial* if it contains a monochromatic 2-fold $(n-k)$-blocking set. Clearly, if we take a 2-fold

$(n-k)$-blocking set and color its points with the same color and give mutually distinct colors for any other point, then we get a proper coloring. This gives a trivial lower bound and we managed to prove that it is sharp in many cases. Let $\theta_i = (q^{i+1}-1)/(q-1)$ denote the number of points in an i-dimensional projective space. We proved that if $n \geq 3$, $1 \leq k < \frac{n}{2}$, and $q \geq 17$ if $k = 1$ and $q \geq 13$ if $k \geq 2$, then the upper chromatic number of $\mathcal{H}(n, n-k, q)$ is $\theta_n - 2\theta_k + 1$. Moreover, we also proved a stability version of the theorem and improved it for the special case when q is prime. Due to the relation with 2-fold blocking sets, we also prove that for $t \leq \frac{3}{8}p + 1$, a small t-fold (weighted) $(n-k)$-blocking set of $\mathrm{PG}(n,p)$, p prime, must contain the weighted sum of t not necessarily distinct $(n-k)$-dimensional subspaces.

◦ Z.L. Blázsik, A. Blokhuis, T. Szőnyi: On the balanced upper chromatic number of finite projective planes. *Manuscript*, under preparation.

We determined the balanced upper chromatic number of the desarguesian projective plane $\mathrm{PG}(2,q)$ for all q. It was motivated by a recent work of Araujo-Pardo, Kiss and Montejano. In general, let $\mathcal{H}$ denote an arbitrary hypergraph. The *balanced upper chromatic number* of $\mathcal{H}$, denoted by $\overline{\chi}_b(\mathcal{H})$, is the largest integer N such that $\mathcal{H}$ admits a proper coloring (see the previous paragraph) using N colors in which the cardinality of any two color classes differs by at most one. We have proved that $\overline{\chi}_b(\mathrm{PG}(2,q)) = \left\lfloor \frac{q^2+q+1}{3} \right\rfloor$. Moreover, we construct some colorings of projective planes represented by cyclic, affine and relative difference sets (or planar functions) giving a lower bound on the balanced upper chromatic number of the right order of magnitude. More precisely, we prove that if $q \equiv 2 \pmod 3$ and Π_q is a projective plane of order q represented by an affine difference set, then $\overline{\chi}_b(\Pi_q) \geq \frac{q^2+2}{3}$, and if $q \equiv 0 \pmod 3$ and Π_q is a projective plane of order q represented by a planar function (or relative difference set), then $\overline{\chi}_b(\Pi_q) = \frac{q^2+q}{3}$. If $q \not\equiv 0 \pmod 3$, then we proved $\frac{q^2+q-1}{3} - \frac{2q^2}{3p}$ (with improvements depending on the remainder of the characteristic p after division by 3). Finally, we used the probabilistic method to show that for an arbitrary projective plane Π_q of odd order q, $\overline{\chi}_b(\Pi_q) \geq \left\lfloor \frac{q^2+q+1}{8\log q+2} \right\rfloor$. As a consequence of these results, we also get a coloring of affine planes represented by planar functions. Recently, Štefko Miklavič has joined this still ongoing project to participate in the research, in particular to analyse the colorings of $\mathrm{AG}(2,5^h)$, which seems to be a special case. He will be a co-author of the final version.

◦ G. Araujo-Pardo, Gy. Kiss, C. Rubio-Montiel, A. Vázquez-Ávila: On chromatic indices of finite affine spaces. *Ars Mathematica Contemporanea*, 16 (2019), 67–79.

◦ G. Araujo-Pardo, Gy. Kiss, C. Rubio-Montiel, A. Vázquez-Ávila: On line colorings of finite projective spaces. *Manuscipt*, arXiv:1702.06769.

In these papers the following problem is considered. Let S be a finite linear space. A *coloring* of S with k colors is an assignment of the lines of S to a set of colors $[k] := \{1,\ldots,k\}$. A coloring of S is called *proper* if any two intersecting lines have different colors. The *chromatic index* $\chi'(S)$ of S is the smallest k such that there exists a proper coloring of S with k colors.A coloring of S is called *complete* if each pair of colors appears on at least one point of S. It is not hard to see that any proper coloring of S with $\chi'(S)$ colors is a complete coloring. The *achromatic index* $\alpha'(S)$ of S is the largest k such that there exists a proper and complete coloring of S with k colors. The *pseudoachromatic index* $\psi'(S)$ of S is the largest k such that there exists a complete coloring (not necessarily proper) of S with k colors. Erdős, Faber and Lovász conjectured that the chromatic index of any finite linear space S cannot exceed the number of its points. The EFL Conjecture was proved for desarguesian projective spaces of any dimension. If Π is an arbitrary (not necessarily desarguesian) finite projective plane of order q, then

$$\chi'(\Pi) = \alpha'(\Pi) = \psi'(\Pi) = q^2 + q + 1,$$

because any two lines of Π have a point in common. The situation is much more complicated in higher dimensional projective spaces, the exact values of the chromatic indices are not known for $n \geq 3$. We studied the achromatic and pseudoachromatic indices of finite spaces. For projective spaces we proved lower estimates on the achromatic index for an infinite series of dimensions and upper estimates on the pseudoachromatic index for any dimension. We proved that the magnitude of the pseudoachromatic index of $\mathrm{AG}(n,q)$ is $q^{1.5n-1}$ for even n, and it is $q^{1.5(n-1)}$ for odd n. Moreover, we proved that the achromatic index of $\mathrm{AG}(n,q)$ is $q^{1.5n-1}$ for even n, and we provided the exact values of both indices in the planar case.

Configurations.

We obtained new results about configurations. Abstract configurations are just regular and uniform hypergraphs. The interesting question is whether they can be realized by using lines, or more precisely line segments and circles, or more precisely circular arcs (or even more generally by segments of conics). Configurations are often beautiful geometric objects that have a nice automorphism group and the proofs often use the Levi graph of the configurations. So, configurations naturally show the links between the topics of our project: graphs, groups and geometries.

◦ G. Gévay: An extension of Miquel's six-circles theorem. *Forum Geometricorum*, Vol 18 (2018) 115–118.

In this work we prove an extension of Miquel's classical six-circles theorem from 6 circles to $2n$ circles ($n \geq 3$).

◦ G. Gévay: Pascal's triangle of configurations. *Discrete Geometry and Symmetry (Eds.: M. Conder, A. Deza and A. Ivič Weiss)*, Springer, 2018, pp. 181–199.

The author introduces an infinite class of configurations which generalize the classical (10_3) Desargues and Danzer's (35_4) configuration. These configurations can be arranged in a triangular array which resembles the classical Pascal triangle also in the sense that it can be recursively generated. They show that each of them expresses an incidence theorem, as well as each can also be represented as a configuration of points and circles.

∘ G. GÉVAY: Resolvable configurations. *Discrete Applied Mathematics*, Volume 266 (2019) 319–330.

Here we introduce the notion of resolvable configurations; they generalize resolvable block designs (which generalize parallelisms in geometric structures). We establish some simple properties of them, on an abstract level. We construct several examples (both sporadic examples and infinite series) of geometric point-line, point-circle and point-conic realizations.

∘ G. GÉVAY, N. BAŠIĆ, J. KOVIČ, T. PISANSKI: Point-ellipse and some other exotic configurations. *Submitted manuscript*, arXiv:1903.06012

We introduce point-ellipse configurations and point-conic configurations. We study some of their basic properties and describe two interesting families of balanced point-ellipse, respectively point-conic 6-configurations. Finally, we investigate a point-ellipse configuration based on the regular 24-cell.

∘ G. GÉVAY, T. PISANSKI: Isometric Miquel configuration of points and circles. *Submitted manuscript*, (2019)

Miquel's classical six-circles theorem leads to a $(8_3, 6_4)$ point-circle configuration. We show that this configuration can be realized by circles of unit radius. Moreover, we prove the following sharpened version of the theorem: If five of the circles of the Miquel configuration have unit radius, then the sixth circle must also have a unit radius.

As a tool, we use the unit-distance version of the incidence graph of the Miquel configuration, which is isomorphic with the skeleton of Kepler's rhombic dodecahedron.

∘ G. GÉVAY, I. HAFNER, T. PISANSKI: Graph of a Six-Cube and Skeleton of a Rhombic Triacontahedron. *Wolfram Demonstrations Project*, Published: September 10, 2019, `http://demonstrations.wolfram.com/GraphOfASixCubeAndSkeletonOfARhombicTriacontahedron`

We demonstrate how the graph of the 6-cube can be built step by step as well as how the skeleton of the rhombic triacontahedron is embedded in this graph.

Integral automorphisms.

We also studied graphs defined on the points of an affine space defined by the analog of the Euclidean distance (so, two points are joined by an edge if and only if their distance is "integral", which means that the square of their usual Euclidean distance is a square of the underlying finite field). This graph has special automorphisms, namely the so-called integral automorphisms which preserve this distance. In this case, we extended earlier work of Kovács and Ruff for higher-dimensional spaces and, in a work in progress, we verified a conjecture of Kiermaier and Kurz. The methods combine (finite) geometric and group theoretic arguments.

◦ I. Kovács, K. Kutnar, J. Ruff, T. Szőnyi: Integral automorphisms of affine spaces over finite fields. *Designs, Codes and Cryptography*, 84 : 1-2 pp. 181-188. , 8 p. (2017)

A permutation of the point set of the affine space is called an integral automorphism if it preserves the integral distance defined among the points. The study of integral automorphisms of finite affine spaces has been initiated by Kiermaier and Kurz. We completed the classification of the integral automorphisms of the affine space of dimension larger than two. We studied the modified version of the problem with non-standard distance functions. This work also motivated the following study, which is still work in progress.

◦ G. Korchmáros, F. Romaniello, T. Szőnyi: Strongly regular graphs from integral point sets in even dimensional affine spaces over finite fields. *Manuscript*, arXiv:1811.06765

We extended the previous work of Kovács, Kutnar, Ruff, Szőnyi in the following direction. We consider the same graph, where the vertices are the points of $\mathrm{AG}(n, q)$ and two distinct vertices are adjacent if their usual squared Euclidean distance is a square in $\mathrm{GF}(q)$, including zero. In 2009, Kurz and Meyer made the conjecture that for n even, this graph is strongly regular. We managed to prove their conjecture, using geometric arguments. More precisely we used the number of points on different types of quadrics over finite fields. Our manuscript is on ArXiv but we forgot to put the acknowledgment on the ArXiv version. It will appear on the published version.

Graphs and geometry.

The next group of papers are related to certain graph parameters and a structure coming from geometry (which may be a finite geometry or a classical one, or even a configuration). In the case of classical geometry, the questions we considered are related to planar graphs or the crossing number. In case of finite geometries we were intersted in the metric dimension and resolving sets of geometric structures.

○ D. Bartoli, T. Héger, Gy. Kiss, M. Takáts: On the metric dimension of affine planes, biaffine planes, and generalized quadrangles. *Australasian Journal of Combinatorics*, 72 (2018), pp. 226–248.

We considered the metric dimension of (the incidence graphs of) particular partial linear spaces. We proved that the metric dimension of an affine plane of order $q \geq 13$ is $3q - 4$ and described all resolving sets of that size if $q \geq 23$. The metric dimension of a biaffine plane of order $q \geq 4$ was shown to fall between $2q - 2$ and $3q - 6$, while for Desarguesian biaffine planes the lower bound is improved to $8q/3 - 7$ under $q \geq 7$, and to $3q - 9\sqrt{q}$ under certain stronger restrictions on q. We determined the metric dimension of generalized quadrangles of order $(s, 1)$, s arbitrary, and derived that the metric dimension of generalized quadrangles of order (q, q), $q \geq 2$, is at least $\max\{6q - 27, 4q - 7\}$, while for the classical generalized quadrangles $W(q)$ and $Q(4, q)$ it is at most $8q$.

○ D. Bartoli, Gy. Kiss, S. Marcugini, F. Pambianco: Resolving sets in higher dimensional projective spaces. *Submitted masucript*, (2017)

Let $\Gamma_{P,\mathcal{H}}(n, q)$ be the point-hyperplane incidence graph of the finite projective space $\mathrm{PG}(n, q)$. The two sets of vertices of this bipartite graph correspond to the points and hyperplanes of $\mathrm{PG}(n, q)$, respectively, and there is an edge between two vertices if and only if the corresponding point is in the corresponding hyperplane. In this paper lower bounds on the size of resolving of $\Gamma_{P,\mathcal{H}}(n, q)$ were proved in a purely combinatorial way. This bound is a generalization of the planar result of Héger and Takáts. Some constructions are also presented. These resolving sets come from lines in higgledy-piggledy arrangement. The proof of our main result shows the existence of six lines in higgledy-piggledy arrangement in $\mathrm{PG}(4, q)$. It is based on estimates of the number of $\mathbb{F}_q$-rational points of suitable algebraic curves.

○ J. Barát, G. Tóth: Improvements on the density of maximal 1-planar graphs. *Journal of Graph Theory*, Volume 88, Issue 1 (2018) 101–109.

A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. A graph, together with a 1-planar drawing is called 1-plane. A graph is maximal 1-planar (1-plane), if we can not add any missing edge so that the resulting graph is still 1-planar (1-plane). Brandenburg et al. showed that there are maximal 1-planar graphs with only $\frac{45}{17}n + O(1) \approx 2.647n$ edges and maximal 1-plane graphs with only $\frac{7}{3}n + O(1) \approx 2.33n$ edges. On the other hand, they showed that a maximal 1-planar graph has at least $\frac{28}{13}n - O(1) \approx 2.15n - O(1)$ edges, and a maximal 1-plane graph has at least $2.1n - O(1)$ edges. We improve both lower bounds to $\frac{20n}{9} \approx 2.22n$.

○ J. Barát, G. Tóth: Improvements on the crossing number of crossing-critical graphs. *Manuscript*, in preparation.

For any graph G, $\mathrm{CR}(G)$ is the crossing number of G. Richter and Thomassen proved in 1993 that any graph G has an edge e such that $\mathrm{CR}(G-e) \geq 2\mathrm{CR}(G)/5$. We improve this bound to roughly $\mathrm{CR}(G)/2$.

Extremal questions.

The investigations in graph theory were extended by extremal questions and the study of graph polynomials, when P. Csikvári joined the project.

○ S. Akbari, P. Csikvári, A. Ghafari, S. Khalashi Ghezelahmad, M. Nahvi: Graphs with integer matching polynomial zeros. *Discrete Applied Mathematics*, **224** (2017), pp. 1-8

In this paper, the authors study graphs whose matching polynomial have only integer zeros. A graph is matching integral if the zeros of its matching polynomial are all integers. the authors characterize all matching integral traceable graphs. They show that apart from $K_7 \setminus E(C_3) \cup E(C_4)$ there is no connected k-regular matching integral graph if $k \geq 2$. It is also shown that if G is a graph with a perfect matching, then its matching polynomial has a zero in the interval $(0, 1]$. Finally, the authors describe all claw-free matching integral graphs.

○ P. Csikvári: Extremal regular graphs: the case of the infinite regular tree. *ArXiv preprint*, 1612.01295

In this paper the authors study the following problem. Let A be a fixed graph, and let $\hom(G, A)$ denote the number of homomorphisms from a graph G to A. Furthermore, let $v(G)$ denote the number of vertices of G, and let $\mathcal{G}_d$ denote the family of d–regular graphs. The general problem studied in this paper is to determine

$$\inf_{G \in \mathcal{G}_d} \hom(G, A)^{1/v(G)}.$$

It turns out that in many instances the infimum is not achieved by a finite graph, but a sequence of graphs with girth (i. e., length of the shortest cycle) tending to infinity. In other words, the optimization problem is solved by the infinite d–regular tree.

The authors prove this type of results for the number of independent sets of bipartite graphs, evaluations of the Tutte-polynomial, Widom-Rowlinson configurations, and many more graph parameters. Our main tool will be a transformation called 2-lift.

○ P. Csikvári: Statistical matching theory. *To appear in the volume Building Bridges II edited by Imre Bárány, Gy. Katona and A. Sali*, Springer

In this paper, the authors survey some recent developments on statistical properties of matchings of very large and infinite graphs. They discuss extremal graph theoretic results like Schrijver's theorem on the number of perfect matchings of regular bipartite graphs and its variants from the point of view of graph limit theory. The authors also study the number of matchings of finite and infinite vertex-transitive graphs.

∘ P. Csikvári, B. Szegedy: On Sidorenko's conjecture for determinants and Gaussian Markov random fields. *ArXiv preprint*, 1801.08425

The authors study a class of determinant inequalities that are closely related to Sidorenko's famous conjecture (also conjectured by Erdős and Simonovits in a different form). The main result can also be interpreted as an entropy inequality for Gaussian Markov random fields (GMRF). The authors call a GMRF on a finite graph G homogeneous if the marginal distributions on the edges are all identical. They show that if G is bipartite then the differential entropy of any homogeneous GMRF on G is at least $|E(G)|$ times the edge entropy plus $|V(G)| - 2|E(G)|$ times the point entropy. They also show that in the case of non-negative correlation on edges, the result holds for an arbitrary graph G. The connection between Sidorenko's conjecture and GMRF's is established via a large deviation principle on high dimensional spheres combined with graph limit theory. It is also observed that the system the authors study exhibits a phase transition on large girth regular graphs. Connection with Ihara zeta function and the number of spanning trees is also discussed.

∘ F. Bencs, P. Csikvári: Note on the zero-free region of the hard-core model. *ArXiv preprint*, 1807.08963

In this paper the authors prove a new zero-free region for the partition function of the hard-core model, that is, the independence polynomials of graphs with largest degree Δ. This new domain contains the half disk

$$D = \left\{ \lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0, |\lambda| \leq \frac{7}{8} \tan\left(\frac{\pi}{2(\Delta - 1)} \right) \right\}.$$

∘ M. Borbényi, P. Csikvári: Counting degree-constrained subgraphs and orientations. *ArXiv preprint*, 1905.06215

The goal of this paper is to advertise the method of gauge transformations (aka holographic reduction, reparametrization) that is the well-known in statistical physics and computer science, but less known in combinatorics. As an application of it they give a new proof of a theorem of A. Schrijver asserting that the number of Eulerian orientations of a d–regular graph on n vertices with even d is at least $\left(\frac{\binom{d}{d/2}}{2^{d/2}} \right)^n$. The authors also show that d–regular graph with even d has always at least as many Eulerian orientations than $(d/2)$–regular subgraphs.

◦ P. Csikvári, A. Imolay: Covers, orientations and factors. *Arxiv preprint*, 1905.06678

Given a graph G with only even degrees let $\varepsilon(G)$ be the number of Eulerian orientations and let $h(G)$ denote the number of half graphs, that is, subgraphs F such that $d_F(v) = d_G(v)/2$ for each vertex v. Recently, M. Borbényi and P. Csikvári proved that $\varepsilon(G) \geq h(G)$ holds true for all Eulerian graphs with equality if and and only if G is bipartite. In this paper the authors give a simple new proof of this fact, and they give identities and inequalities for the number of Eulerian orientations and half graphs of a 2-cover of a graph G.

Finite geometrical questions.

Some of our work was related groups and geometries, but essentially they answered questions in geometry. As indicated in our proposal, we also considered questions in geometry (without immediate applications to graph problems). They are either interesting on their own right or have the potential to apply them for graph problems. Some of them have a strong algebraic connection with loops, some with bilinear forms (or geometrically with polar spaces). In most cases the papers use algebraic machinery coming from the theory of algebraic curves over finite fields, special equations or polynomials over finite fields.

◦ G. Korchmáros, G. P. Nagy: Group-labeled light dual multinets in the projective plane. *Discrete Mathematics*, 341 (2018), no. 8, 2121–2130. arXiv:1710.00172

In this paper, we investigate light dual multinets labeled by a finite group in the projective plane $\mathrm{PG}(2, K)$ defined over a field K. In contrast to dual multinets, light dual multinets have lines intersecting each of the three components in $r > 1$ points. Due to the connection between embedded dual multinets and completely reducible elements of pencils of curves, this light (dual) multinet setting is very natural. The known constructions were given by Bartz and Yuzvinsky in 2013. We presented two infinite classes of new examples, and a sporadic example which embeds a nonassociative quasigroup of order 18. The two new classes have been named "triangular" and "conic-line" type, as the point sets of the light dual multinets are contained in a reducible cubic curve. Moreover, under some conditions on the characteristic of K, we classify group-labeled light dual multinets with lines of length least 9, by showing that they are either triangular or of conic-line type of our construction.

◦ N. Bogya, G. P. Nagy: Light dual multinets of order six in the projective plane. *Acta Mathematica Hungarica*, Article in Press, 2019. arXiv:1810.00416

We continued the study of light dual multinets in the projective plane $\mathrm{PG}(2, K)$. The aim of this paper was twofold: First we classifies all abstract light dual multinets of order 6 which have a unique line of length at least two. Then we classified the weak projective embeddings of these objects in projective planes over fields of characteristic zero. For the latter we presented a computational algebraic method for the study of weak projective embeddings of finite point-line incidence structures.

◦ G. Korchmáros, G. P. Nagy, P. Speziali: Hemisystems of the Hermitian Surface. *Journal of Combinatorial Theory, Series A*, 165 (2019), 408–439. arXiv:1710.06335

We presented a new method for the study of hemisystems of the Hermitian surface $\mathcal{U}_3$ of $\mathrm{PG}(3, q^2)$. A hemisystem is a set H of generators of $\mathcal{U}_3$ such that each point of $\mathcal{U}_3$ is covered by $q + 1)/2$ elements of H. Construction of hemisystems on $\mathcal{U}_3$ of $\mathrm{PG}(3, q^2)$, as well as in any generalized quadrangle of order (q^2, q), is a relevant issue because hemisystems give rise to important combinatorial objects which have been under investigation for many years, such as strongly regular graphs, partial quadrangles and 4-class imprimitive cometric Q-antipodal association schemes that are not metric. Our basic idea is to represent generator-sets of $\mathcal{U}_3$ by means of a maximal curve naturally embedded in $\mathcal{U}_3$ so that a sufficient condition for the existence of hemisystems may follow from results about maximal curves and their automorphism groups. In this paper we obtain a hemisystem in $\mathrm{PG}(3, p^2)$ for each p prime of the form $p = 1 + 16n^2$ with an integer n. Since the famous Landau's conjecture dating back to 1904 is still to be proved (or disproved), it is unknown whether there exists an infinite sequence of such primes. The scarcity of such primes seems to confirm that hemisystems of $\mathcal{U}_3$ are rare objects.

◦ A. Blokhuis, I. Kovács, G. P. Nagy, T. Szőnyi: Inherited conics in Hall planes. *Discrete Mathematics*, 342 (2019), no. 4, 1098–1107. arXiv:1805.09984

We studied the combinatorial properties of inherited conic of the finite Hall plane. In general, the existence of ovals and hyperovals is an old question in the theory of non-Desarguesian planes. The aim of the paper is to describe when a conic of $\mathrm{PG}(2, q)$ remains an arc in the Hall plane obtained by derivation. This is a general idea for constructing interesting subsets of the Hall plane. The key ingredient of the proof was an old lemma by Segre and Korchmáros on Desargues configurations with perspective triangles inscribed in a conic. Based on this lemma, we have been able to give an algebraic method to determine the number of collinear triples in the inherited conic. The more precise combinatorial behaviour depends heavily on the characteristic and the affine position of the underlying conic. In particular, in many cases the combinatorial properties of the resulting pointset are very much different from those of pointsets in $\mathrm{PG}(2, q)$.

◦ T. Szőnyi, Zs. Weiner: Stability of $k \mod p$ multisets and small weight codewords of the code generated by the lines of $\mathrm{PG}(2, q)$. *Journal of Combinatorial Theory, Series A*, Volume 157, July 2018, Pages 321-333.

This work shows that any multiset of points of $\mathrm{PG}(2, q$ with at most roughly $cq\sqrt{q}$ non-k mod

p secants can be obtained from a k mod p multiset by modifying a small number of points. Then these results are used to show that small weight codewords in the code generated by the lines of PG2, q) are linear combinations of lines. The prime case is surprisingly difficult, there are codewords of weight $3p$, which are not linear combinations of lines. However, it is true that the support of a small weight codeword can be covered by few lines.

∘ J. De Beule, J. Demeyer, S. Mattheus, P. Sziklai: On the cylinder conjecture. *Designs, Codes, and Cryptography*, 87 (2019), no. 4, 879-893.

In this paper, the authors associate a weight function with a set of points satisfying the conditions of the cylinder conjecture. Then they derive properties of this weight function using the Rédei polynomial of the point set. Using additional assumptions on the number of non-determined directions, together with an exhaustive computer search for weight functions satisfying particular properties, they prove a relaxed version of the cylinder conjecture for $p \leq 13$. This result also slightly refines a result of Sziklai on point sets in AG$(3, p)$.

∘ D. Mezőfi, G. P. Nagy: UnitalSZ. *GAP package*, hosted on GitHub Pages, `https://nagygp.github.io/UnitalSZ/README.html`

We took part in the development of the GAP package 'UnitalSZ' containing methods for abstract unitals as block designs. The package contains methods for automorphisms and isomorphisms and for the embeddings of unitals in the finite Desarguesian projective plane which is considered as a difficult and important problem of finite geometry. Moreover, the package has functions for constructing unitals and some libraries of unitals of small order. With the help of this package we found a simple proof for the uniqueness of the embedding of Hermitian unitals in Desarguesian projective planes.

Graphs with strong transitivity properties.

Graphs with strong transitivity properties were mainly studied by our Slovenian colleagues. However, the Hungarian researchers also contributed to results on important classes of such graphs, for example the generalized Petersen graphs. This contribution also includes the introduction of a new regularity property for graphs, the edge-girth regularity property.

∘ J. Barát: Decomposition of cubic graphs related to Wegner's conjecture. *Discrete Mathematics*, Volume 342, Issue 5 (2019) 1520–1527.

Thomassen formulated the following conjecture: Every 3-connected cubic graph has a red–blue vertex coloring such that the blue subgraph has maximum degree 1 (that is, it consists of a matching and some isolated vertices) and the red subgraph has minimum degree at least 1 and contains no 3-edge path. We prove the conjecture for Generalized Petersen graphs. We indicate that a coloring with the same properties might exist for any subcubic graph. We confirm this statement for all subcubic trees.

◦ R. Jajcay, Gy. Kiss, Š. Miklavič: Edge-girth-regular graphs. *European Journal of Combinatorics*, 72 (2018), 70–82.

In this paper, we introduced new type of regular graphs. An edge-girth-regular (v, k, g, λ)-graph is a k-regular graph of order v and girth g in which every edge is contained in λ distinct g-cycles. This concept is a generalization of the well-known (v, k, λ)-edge-regular graphs (that count the number of triangles) and appears in several related problems such as Moore graphs and Cage and Degree/Diameter Problems. All edge- and arc-transitive graphs are edge-girth-regular as well. We derived a number of basic propertiesof edge-girth-regular graphs, systematically considered cubic and tetravalent graphs from this class, and introduced several constructions that produce infinite families of edge-girth-regular graphs. We also exhibited several surprising connections to regular embeddings of graphs in orientable surfaces.

Miscellaneous results.

Let us finally collect here those of our works that are somewhat loosely connected to graphs and geometries but fit in the general framework of our research. Among the problems considered, some are related to hypergraphs or Latin squares, graphical Frobenius representations of non-abelian groups and doubly transitive sets of even permutations. Last but not least, participants of the project published a book on finite geometries. The first papers are related to the famous Ryser conjecture for r-partite r-uniform hypergraphs.

◦ A. Abu-Khazneh, J. Barát, A. Pokrovskiy, T. Szabó: A family of extremal hypergraphs for Ryser's conjecture. *Journal of Combinatorial Theory, Series A*, Volume 161 (2019) 164–177.

Ryser's Conjecture states that for any r-partite r-uniform hypergraph, the vertex cover number is at most $r - 1$ times the matching number. This conjecture is only known to be true for $r \leq 3$ in general and for $r \leq 5$ if the hypergraph is intersecting. There has also been considerable effort made for finding hypergraphs that are extremal for Ryser's Conjecture, i.e. r-partite hypergraphs whose cover number is $r - 1$ times its matching number. Aside from a few sporadic examples, the set of uniformities r for which Ryser's Conjecture is known to be tight is limited to those integers for which a projective plane of order $r - 1$ exists. We produce a new infinite family of r-uniform hypergraphs extremal to Ryser's Conjecture, which exists whenever a projective plane of order $r-2$ exists. Our construction is flexible enough to produce a large number of non-isomorphic extremal hypergraphs.

◦ J. Barát: Intersecting hypergraphs with maximum covering number. *Manuscript*, in preparation.

Erdős and Lovász proved that an r-uniform intersecting hypergraph with maximum covering number, that is $\tau(H) = r$, must have at least $\frac{8}{3}r - 3$ edges. There has been no improvement on this lower bound for 45 years. We try to understand the reason by studying some small cases to see whether the truth lies very close to this simple bound. We obtain the following results: $q(3) = 6$, $q(4) = 9$, $q(5) = 13$, where $q(r)$ denotes the minimum number of edges in an intersecting r-uniform hypergraph. We use both theoretical arguments and computer searches. One standard tool is nauty by Brendan McKay. In the footsteps of Erdős and Lovász, we also consider the special case, when the hypergraph is part of a finite projective plane. In case of intersecting r-partite hypergraphs, Ryser's conjecture claims the maximum covering number can be $r - 1$. Recently, there has been accelerated interest in finding sharp examples. Király and Tóthmérész conjectured the following t-intersecting generalisation. If H is an r-partite hypergraph and any two hyperedges intersect in at least t vertices, then $\tau(H) \leq r - t$. Inspired by this question, we try to find 2-intersecting r-uniform hypergraphs with maximum covering number, that is $\tau(H) = r - 1$. We found that some biplanes are extremal, but not all of them.

◦ J. Barát: On the number of edges in a K_5-minor-free graph of given girth. *Manuscript*, in preparation.

We know that every planar graph has a vertex of degree at most five, whence it follows that it is 6-list colorable. In this work, we consider a graph family close to planar graphs, namely, K_5-minor-free graphs. Many list coloring problems involve the girth of the graph, which motivates the study of K_5-minor-free graph of given girth. We determine the maximal number of edges a K_5-minor-free graph of girth 4 or 5 may have, and we also study the extremal examples.

◦ J. Barát, Z. L. Nagy: Transversals in generalized Latin squares. *Ars Mathematica Contemporanea*, Vol. 16 (2019) 39–47.

We are seeking a sufficient condition that forces a transversal in a generalized Latin square. A generalized Latin square of order n is equivalent to a proper edge-coloring of $K_{n,n}$. A transversal corresponds to a multicolored perfect matching. Akbari and Alipour defined $l(n)$ as the least integer such that every properly edge-colored $K_{n,n}$, which contains at least $l(n)$ different colors, admits a multicolored perfect matching. They conjectured that $l(n) \leq n^2/2$ if n is large enough. In this note we prove that $l(n)$ is bounded from above by $0.75n^2$ if $n > 1$. We point out a connection to anti-Ramsey problems. We propose a conjecture related to a well-known result by Woolbright and Fu, that every proper edge-coloring of K_{2n} admits a multicolored 1-factor.

◦ G. Korchmáros, G. P. Nagy: Graphical Frobenius representations of non-abelian groups.

Submitted, arXiv:1909.03690

A group G has a Frobenius graphical representation (GFR) if there is a simple graph H whose full automorphism group is isomorphic to G and it acts on vertices as a Frobenius group. In particular, any group G with GFR is a Frobenius group and H is a Cayley graph. The existence of an infinite family of groups with GFR whose Frobenius kernel is a non-abelian 2-group has been an open question, asked by Doyle, T. W. Tucker, and M. E. Watkins (2018). In this paper, we gave a positive answer by showing that the Higman group $A(f, q_0)$ has GFR for an infinite sequence of integers f and q_0.

∘ G. P. Nagy: Doubly transitive sets of even permutations. *Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica*, Number 1(80) (2016), 78–82

This paper gives a partial answer to a question by Károlyi on the existence of sharply 2-transitive sets in the alternating group of degree 0 or 1 modulo 4. We show that for infinitely many integers $n \equiv 0, 1 \pmod 4$, A_n does contain a sharply 2-transitive set.

∘ Gy. Kiss, T. Szőnyi: Finite Geometries. *CRC Press - Taylor & Francis Group, Boca Raton*, (2019) 338 p.

Tamás Szőnyi and György Kiss finished their book *Finite Geometries* published by CRC Press. The material of the book was used in finite geometry courses both at Eötvös Loránd University and also at the University of Primorska. (Let us note that the acknowledgement of the support of the present project is put in the preface.)

Summary of other activities related to the project.

According to our plans, we organized three workshops related to the project. The first one was held in Pécs (2016) with roughly 35 participants, the second and third workshops were held in Szeged (2017, 2019) with roughly 40 and 70 participants, respectively. In all cases, there were one or two Slovenian colleagues among the invited speakers and in total there were 6-10 Slovenian participants. Besides these workshops, our results were also presented at international conferences. The Hungarian participants of the project attended about 6-7 international conferences each year, in total we supported more than 30 conference participation (most with conference talk) for them.