In the project K113047, according our research plan, we have done research and established a number of significant results in the following areas:

- Set-theoretic topology
- Infinite combinatorics
- Descriptive set theory and real analysis
- Model theory and philosophy of mathematics

We presented our results in 53 publications, almost all of which appeared or will appear in the leading international journals of these fields (6 of these papers have been submitted but not accepted as yet). The quality of our results is well demonstrated by the fact that 20 of our papers appeared in Q1 and 14 in Q2 journals.

Our research group consisted of the PI and 8 participants, moreover 2 PhD students. Both of them received their degrees in the course of our project and contributed significantly to our results. M. Vizer left the project on 2015-09-01 and on the same date D. Soukup joined it. Also, A. Joó joined our project on 2017-09-01. Members of our group participated at a large number of international conferences, five of us (Elekes, Juhász, Sági, L. Soukup and D. Soukup) as plenary and/or invited speakers at many of these. We now give an overview of our results.

I. Set-theoretic topology

In 4. we started to study "neighbourhood assignment versions" of various topological cardinal functions. For instance, the neighbourhood assignment version $pd(X)$ of the density function $d(X)$ is defined as the smallest cardinal κ such that for any neighbourhood assignment of X there is a set of cardinality κ that meets every member of the assignment.

The first main theorem of 4. says that the following three statements are equivalent:
(1) Every Hausdorff space X satisfies $pd(X) = d(X)$.
(2) Every 0-dimensional Hausdorff space X satisfies $pd(X) = d(X)$.
(3) Every limit cardinal is strong limit.
This solved two problems that were raised by Banakh and Ravsky.

In particular, this implies that the existence of (nice) spaces with $pd(X) < d(X)$ is independent of ZFC. For brevity, we call such a space X a *pd-example.* We showed that every pd-example X has an open subspace Y that is also a pd-example and has the additional property that all non-empty open sets in Y have the same size. We call such a pd-example *canonical.*

The second main result of 4. says that the following three statements are *equiconsistent*:
(I) Shelah's Strong Hypothesis fails.
(II) There is a 0-dimensional Hausdorff *canonical* pd-example of regular cardinality.
(III) There is any *canonical* pd-example of regular cardinality.
This explains why it is much easier to find canonical pd-examples of singular cardinality than those of regular cardinality.

We also proved that the classical inequality $|X| \leq 2^{2^{d(X)}}$ for Hausdorff spaces can be improved to $|X| \leq 2^{2^{pd(X)}}$. The question if the inequality $w(X) \leq 2^{d(X)}$ for *regular* spaces can be improved to $w(X) \leq 2^{pd(X)}$ remains open, although we could prove several partial affirmative results pointing in this direction.

We proved in 27. that all our pd-examples from 4., whose existence is equivalent to the existence of a singular cardinal that is not strong limit, may be improved to be either (locally) connected or topological groups, without any further assumptions. However, we also proved that the existence of a pd-example that is both connected and Tychonov, in particular of a connected topological group, already does require some further assumptions. They do exist if there is a singular cardinal $\geq \mathfrak{c}$ that is not strong limit and this condition is also necessary if Shelah's strong hypothesis holds. Moreover, it actually yields pd-examples that are even topological vector spaces.

In 5. we investigated anti-Urysohn (in short: AU) spaces, i.e. Hausdorff spaces in which any two regular closed sets meet. We proved that for every infinite cardinal κ there is an AU of cardinality κ. This is interesting even for the case $\kappa = \omega$, as our construction seems to provide the simplest and, in some sense, strongest example of a countable connected Hausdorff space. (Note that an AU space is trivially connected.) We could also produce locally countable AU spaces of size κ for every infinite $\kappa \leq 2^{\mathfrak{c}}$. This is optimal because it is easy to show that any locally countable and connected Hausdorff space has cardinality $\leq 2^{\mathfrak{c}}$.

We also studied in 5. a strengthening of anti-Urysohn called strongly anti-Urysohn (in short: SAU). A space is SAU if it is Hausdorff with at least two non-isolated points and any two *infinite closed* sets in it meet. We constructed several consistent examples of SAU spaces with no isolated points. (Clearly, such a SAU is AU.) The question if a SAU exists in ZFC, however, remains open. We proved that if X is SAU then $\mathfrak{s} \leq |X| \leq 2^{2^{\mathfrak{c}}}$, where $\mathfrak{s}$ is the splitting number. We could show that the lower bound $\mathfrak{s}$ is (consistently) sharp but we conjecture that the upper bound $2^{2^{\mathfrak{c}}}$ can be improved. In fact, all our (consistent) examples are of cardinality $\leq \mathfrak{c}$.

In 15. we introduced and studied two new classes of spaces: almost discretely Lindelöf (ADL) spaces and weakly linearly Lindelöf (WLL) spaces. ADL spaces are those in which any discrete subset can be covered by a Lindelöf subspace, while a space is WLL if every system of open sets of uncountable regular size in it has a complete accumulation point. ADL and weakly Lindelöf spaces, as well as linearly Lindelöf spaces, are WLL. The main result of 15. is that the WLL property implies the much stronger Lindelöf property in the important class of monotonically normal spaces.

In 25. we solved a problem raised in 15. by proving that every ADL first countable regular space has cardinality at most continuum. This is a significant improvement on Arhangel'skii's celebrated result concerning the cardinality of first countable Lindelöf spaces.

Homogeneous compacta are ubiquitous throughout mathematics. Examples include all compact connected manifolds and compact topological groups. Hence results that concern homogeneous compacta are of general interest. In 28. we proved that every (infinite) homogeneous compactum that is either the union of countably many dense or of finitely many arbitrary *countably tight* subspaces has cardinality continuum. This is a significant improvement on de la Vega's similar result for countably tight homogeneous compacta and required the use of some non-trivial new techniques and results.

A space X is called densely k-separable if every dense subspace of X has a *σ-compact* dense subset. In 35. we answered a problem raised by J. van Mill by proving that any densely k-separable compactum X is actually densely separable, i.e. every dense subspace of X has a *countable* dense subset.

The G_δ-modification X_δ of a space X is the space on the same underlying set generated by the collection of all G_δ subsets of X. Bella and Spadaro recently investigated the connection between the values of various cardinal functions taken on X and X_δ, respectively. They raised the following two problems: Is $t(X_\delta) \le 2^{t(X)}$ true for every (compact) T_2 space X? In 36. we answered both questions: In the compact case affirmatively and in the non-compact case negatively. In fact, we even showed that it is consistent with ZFC that no upper bound exists for the tightness of the G_δ-modifications of countably tight, even Fréchet spaces.

In 42. two of the most important cardinal functions, namely the weight and the cellularity appear in a new role. The following are the main results of 42.:

(A) Let κ be an uncountable regular cardinal. Then any dense-in-itself Tychonov space X with $\widehat{c}(X) = \kappa$ has a nowhere constant continuous image Y of weight $w(Y) \le \kappa$. Moreover, $\le$ can be replaced by $<$ exactly iff no κ-Suslin line exists.

(B) If κ and X are as in 1. then X has a pseudo-open continuous image Y of weight $w(Y) \le 2^{<\kappa}$. Moreover, Martin's axiom implies that this result is sharp.

A continuous map $f : X \to Y$ is nowhere constant (resp. pseudo-open) if the pre-image of every point of Y (resp. every nowhere dense subset of Y) is nowhere dense in X.

If P is a property of topological spaces, a map $f : X \to Y$ is called *P-preserving* if, for every subspace $A \subset X$ with property P, its image $f(A)$ also has property P. As continuous maps are both compactness- and connectedness-preserving, a natural question is under what conditions is such a map continuous. This has been thoroughly investigated by numerous authors. Our main result in 24. is the following surprising theorem: Any non-trivial product function, i.e. one having at least two non-constant factors, that has connected domain, T_1 range, and is connectedness-preserving must actually be continuous. We present examples which show that the analogous statement badly fails if we replace in it the occurrences of "connected" by "compact". We also presented several interesting results and examples concerning maps that are compactness-preserving and/or continuum-preserving. In particular, we constructed under the continuum hypothesis a locally connected continuum that is both compactness-preserving and continuum-preserving but not connectedness-preserving.

For more than two decades now, we have invested a lot of effort in the study of resolvability, i.e. in the question of how many disjoint dense sets exist in certain topological spaces.

One of the oldest and toughest open questions concerning resolvability is whether crowded pseudocompact spaces are resolvable. We proved in 26. that any such space is even $\mathfrak{c}$-resolvable, provided that every disjoint collection of open subsets in it has size at most a finite successor of $\mathfrak{c}$. Moreover, if the continuum hypothesis fails then in this statement "finite" can be replaced by "countable". We also proved that it is consistent with ZFC that *every* such space is $\mathfrak{c}$-resolvable, while the value of $\mathfrak{c}$ can be arbitrarily big. These results are also of interest because even for the much smaller class of crowded countably compact Tychonov spaces it is an open question whether they are $\mathfrak{c}$-resolvable.

The method of proof of our results is of particular interest. It hinges on the existence of a "Bernstein type" coloring property of the family of all Cantor subsets of generalized Baire spaces.

Another outstanding problem in this area, due to Malychin, asks how resolvable is a regular Lindelöf space in which all non-empty open sets are uncountable. In 43. we could extend our earlier deep result proving the ω-resolvability of regular spaces of countable extent and uncountable dispersion character to the much wider class of (countable extent)-generated spaces. This extension is highly non-trivial and greatly improves not only our earlier result but also some recent result of Filatova and Osipov.

Any crowded space X becomes ω-resolvable in the generic extension of the ground model obtained by adding $|X|$ many Cohen reals. We call the space X *monotonically* ω_1*-resolvable* if there is a function $f : X \to \omega_1$ such that $\{x \in X : f(x) \geq \alpha\}$ is dense in X for all $\alpha < \omega_1$. We proved in 18. that for a T_1-space X being monotonically ω_1-resolvable is equivalent with both becoming ω_1-resolvable in some c.c.c-generic extension and becoming ω_1-resolvable after adding ω_1 Cohen reals.

Moreover, we showed that (1) if X is c.c.c and $\omega < \Delta(X) \leq |X| < \aleph_\omega$, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open in } X\}$, then X is monotonically ω_1-resolvable, (2) assuming the existence of a measurable cardinal it is consistent that there is a space Y with $|Y| = \Delta(Y) = \aleph_\omega$ which is not monotonically ω_1-resolvable.

Another topic that has been in the center of our investigations is the study of cardinal sequences of compact scattered (CS) spaces. We characterized these of length $\leq \omega_1$ in ZFC and of length $< \omega_2$ under GCH. However, above ω_2 the problem becomes much harder. In 47. we proved that if GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal-preserving generic extension $\mathfrak{c} = \lambda$ and every sequence $\langle \kappa_\alpha : \alpha < \eta \rangle$ of infinite cardinals of length $\eta < \omega_3$ such that $\kappa_\alpha \leq \lambda$ for all $\alpha < \eta$ and $\kappa_\alpha = \omega$ when $\text{cf}(\alpha) = \omega_2$ is the cardinal sequence of some CS space. We also proved that for each uncountable cardinal λ it is consistent that $\langle \omega \rangle_\alpha {}^\frown \langle \lambda \rangle_\beta$ is a cardinal sequence whenever $\alpha, \beta < \omega_3$ and $cf(\alpha) < \omega_2$. Here $\langle \omega \rangle_\alpha {}^\frown \langle \lambda \rangle_\beta$ is the sequence of length $\alpha + \beta$ whose first α terms equal ω and the next ones equal λ.

In 46. we obtained some results for much longer sequences. We could prove that it is consistent for any regular cardinal κ and ordinal $\eta < \kappa^{++}$ that 2^κ is as large as you wish and every sequence $\langle \kappa_\alpha : \alpha < \eta \rangle$ with $\kappa \leq \kappa_\alpha \leq 2^\kappa$ and $\kappa_\alpha = \kappa$ for $\text{cf}(\alpha) < \kappa$ is the cardinal sequence.

In 13. we analyzed the following problem: given two chip-distributions on a digraph, decide whether the first one can be reached from the second one by playing a legal chip-firing game. We showed that this problem can be decided in polynomial time for Eulerian digraphs, even if the digraph has multiple edges. We also showed that if the target distribution is recurrent, that is, it is reachable from itself, then the problem can be decided in polynomial time on a general digraph.

II. Infinite combinatorics

Answering a 1978 question of R. Rado, we showed in 6. that every finite-edge colored complete graph on ω can be partitioned into disjoint monochromatic paths of different colors. In 9. we considered numerous variations of this theme, hypergraphs instead of graphs, the case of ω_1 as the vertex set, etc.

In 20. we used a general method based on trees of elementary submodels to give highly simplified proofs of numerous results in infinite combinatorics. While countable elementary submodels have been employed in such settings already, we significantly broadened this framework by developing the corresponding technique for countably closed models of size continuum. We applied our method to prove theorems on paradoxical decompositions of the plane, coloring sparse set systems, graph chromatic number and constructions from set-theoretic topology.

N. Hindman, I. Leader and D. Strauss proved the consistent existence of a finite coloring of $\mathbb{R}$ such that for no infinite $X \subset \mathbb{R}$ is the sumset $X + X$ monochromatic. In 45. we proved a consistency result going in the opposite direction: We showed that, under a certain set-theoretic assumption involving large cardinals, and whose consistency was established by Shelah, for any finite coloring $c : \mathbb{R} \to r$ there is an infinite $X \subseteq \mathbb{R}$ such that c is constant on $X + X$.

In 39. we showed that if D is a tournament of arbitrary size then D has finite strong components after reversing a locally finite sequence of cycles. We also proved that any tournament can be covered by two acyclic sets after reversing a locally finite sequence of cycles. This provides a partial solution to a conjecture of S. Thomassé.

A linear order L is called strongly surjective if L can be mapped onto any of its suborders in an order preserving way. In 32. we proved various results on the existence and non-existence of uncountable strongly surjective linear orders, answering questions of Camerlo, Carroy and Marcone.

Another area of our research concerns finding suitable extensions or analogues of results of finite combinatorics for the infinite case. It follows from a theorem of Lovász, that if D is a finite digraph with $r \in V(D)$, then there is a spanning subdigraph E of D such that for every vertex $v \neq r$ the following three quantities are equal: the local connectivity from r to v in D, the local connectivity from r to v in E, the indegree of v in E. The main result of 40. is a generalization of this theorem to countable digraphs in an "Erdős-Menger-like" way. We construct a spanning subdigraph E of D such that for every $v \neq r$ it contains a strongly maximal internally disjoint system P_v of $r \to v$ paths of D where the ingoing edges of v in E are exactly the last edges of the paths in P_v. The strong maximality of P_v in D means that for every internally disjoint system Q of $r \to v$ paths in D we have $|Q \setminus P_v| \leq |P_v \setminus Q|$ which implies that P_v is big in D in the Erdős-Menger-sense.

Nash-Williams proved, that for an undirected graph G the set $E(G)$ of its edges can be partitioned into cycles if and only if every cut has either even or infinite number of edges. C. Thomassen gave a simpler proof for this and conjectured the following directed analogue: the edge-set of a digraph can be partitioned into directed cycles if and only if for each subset of the vertices the cardinality of the ingoing and the outgoing edges are equal. In 41. we proved this conjecture.

III. Descriptive set theory and real analysis

In 7. we characterized the possible order types of linear orderings consisting of Baire class 1 functions (i.e. pointwise limits of continuous functions), ordered by the pointwise ordering. This solved an open problem posed by M. Laczkovich in the 70s. Using this characterization we were able to answer all the open problems in this area.

In 8. we continued our work on the regularity properties of Christensen's notion of Haar null sets. We proved that certain naive modifications of this notion behave badly because the collections of sets obtained in this way are not closed under unions. This generalizes a result of Elekes and Steprans.

Kechris and Louveau proved that each real-valued bounded Baire class 1 function defined on a compact metric space can be written as an alternating sum of a decreasing countable transfinite sequence of upper semi-continuous functions. Moreover, the length of the shortest such sequence is essentially the same as the value of a certain natural rank they defined. In 16. we generalized this result to arbitrary Polish spaces. Also, using the topology refinement method we could prove analogous statements about Baire class ξ functions.

In 29. we investigated the notion of Borel chromatic numbers of Borel graphs, i.e., the definable analogs of the classical notions. The main result is that, unlike in the case of uncountable Borel chromatic numbers, it is impossible to find a simple collection of Borel graphs so that a Borel graph has contably infinite Borel chromatic number if and only if it contains a homomorphic copy of one of them. This result, besides answering several open questions and having importance in the theory of Borel graphs, may be used to exclude the existence of certain graph coloring algorithms.

In 31. we proved that there exists a single element basis for graphs of Borel chromatic number at least 3. This, together with the results of 29. completely describes which Borel chomatic numbers of Borel graphs may be characterized in terms of simple bases.

In 30. we analyzed the embeddability relation defined by S. Solecki between graphs of functions of different complexity classes. We could show that, for every $\xi \geq 1$, there is no maximal element with respect to this relation in the family of Baire class ξ functions. Moreover there is a continuum sized antichain in each such family.

In 33. we used Haar null sets to investigate the structure of the random element in various homeomorphism groups. We gave a characterization of the non-Haar null conjugacy classes for the groups of order-preserving homeomorphisms of the interval and the circle. We also showed that, apart from the classes of the multishifts, every conjugacy class is Haar null in the group of the unitary transformations of the separable infinite-dimensional Hilbert space.

Analogous questions were investigated in 34. for automorphism groups of countable first-order structures, hence developing a dual theory to that of Kechris and Rosendal. We generalized theorems of Dougherty and Mycielski about S_∞ to arbitrary automorphism groups of countable structures, isolating a new model theoretic property, the Cofinal Strong Amalgamation Property. A complete description of the non-Haar null conjugacy classes of the automorphism groups of $(\mathbb{Q}, <)$ and of the random graph was given. We proved that every non-Haar null class contains a translated copy of a non-empty portion of every compact set. As an application we affirmatively answered the question whether these groups can be written as the union of a meager and a Haar null set.

We also used the notion of Haar null sets in 51. to investigate the structure of random elements of the automorphism group of the rational numbers. we gave a complete description of the size of the conjugacy classes of the group with respect to this notion. In particular, we showed that there exist continuum many non-Haar null conjugacy classes, illustrating that the random behaviour is quite different from the Baire category case.

In 37, we studied the notion of Haar meager sets in (not necessarily locally compact) Polish topological groups, a dual notion to Haar null sets. We answered Darji's problem, one of the most important questions of this field, by constructing a set that is Haar meager but not strongly Haar meager, despite the fact that several earlier results suggested that they coincide.

In 38. we investigated the so called cardinal invariants of the Hausdorff measures, showing how they fit into the famous Cichon Diagram. We solved a problem of Fremlin by proving that a certain inequality in this extended diagram can be strict. We also constructed by forcing a model in which there is an ordering of the reals such that all proper initial segments are Lebesgue null but the analogous statement fails for the $1/2$-dimensional Hausdorff measure. This answers a question of Humke-Laczkovich. Finally, we applied our methods to provide a non-homogeneus idealized forcing, thus answering a question of Zapletal.

Duparc introduced a two-player game for a self-map f of the Baire space in which Player II has a winning strategy iff f is Baire class 1. In 44. we defined an analogous game that can be used to characterize Baire class 1 functions between any two Polish spaces. Then this was used to reprove a known result in the theory of first return recoverable functions.

In 48. we established a graph theoretic dichotomy which enabled us to give a new, simple proof of the following celebrated result of Harrington, Marker and Shelah: Any Borel partially ordered set either can be covered countably many Borel chains or contains a perfect anti-chain. Moreover, our graph theoretic approach made it possible to generalize the theorem for higher projective classes.

IV. Model theory and philosophy of mathematics

Let $\mathcal{D}$ be a complexity class. A countable first order structure is defined to be $\mathcal{D}$-presented iff all of its basic relations and functions are in $\mathcal{D}$. We showed in 17. that if T is a first order theory with at least one uncountable Stone space then T has a countable model not isomorphic to any $\mathcal{D}$-presented one. We also show that there is a countable $\aleph_0$-categorical structure in a finite language which is not isomorphic to any $\mathcal{D}$-presented structure; in addition, there exists a consistent first order theory in a finite language that does not have $\mathcal{D}$-presented models, at all. Our proofs are purely model theoretic and do not involve any nontrivial recursion theoretic notion or construction.

The Craig interpolation property has been under thorough investigation ever since Craig proved that it holds for usual first order logic. Related problems have been intensively studied in the literature of algebraic logic. It turned out that interpolation properties of different logics are strongly related to various amalgamation properties of certain classes of algebras. In 22. we gave an algebraic characterization of a local version of Craig's interpolation theorem in terms of a superamalgamation property.

In 10. and 11. we continued the investigation of the modal logic of Bayesian belief. We took the more general Jeffrey formula as a conditioning device and studied the corresponding modal logics that we call Jeffrey logics, focusing mainly on the countable case. The containment relations among these modal logics were determined and it was shown that the logic of Bayes and Jeffrey updating are very close. It was shown that modal logic of Bayesian belief revision determined by probabilities on a finite or countably infinite set of elementary propositions is *not finitely axiomatizable*. The significance of this result is that it clearly indicates that axiomatic approaches to belief revision might be severely limited. The infinite case remained open in this paper. However, in 11. we proved that the modal logic of Bayesian belief revision determined by standard Borel spaces is also not finitely axiomatizable, solving the problem for a large class of infinite probability spaces.