Final Report on the Project Geometry of Manifolds (NKFI-ID: K112703)

B. Csikós and M. Horváth studied geometric invariants of tubes about curves in a Riemannian manifold. H. Hotelling proved that in the n-dimensional Euclidean or spherical space, the volume of a tube of small radius about a curve depends only on the length of the curve and the radius. A. Gray and L. Vanhecke extended Hotelling's theorem to rank one symmetric spaces computing the volumes of the tubes explicitly in these spaces. B. Csikós and M. Horváth [1] generalized these results by showing that every harmonic manifold has the above tube property. They computed the volume of tubes in the Damek–Ricci spaces. They showed that if a Riemannian manifold has the tube property, then it is a 2-stein D'Atri space. They also proved that a symmetric space has the tube property if and only if it is harmonic. These results answered some questions posed by L. Vanhecke, T. J. Willmore, and G. Thorbergsson, and supported the conjecture that manifolds with the tube property are exactly the harmonic manifolds.

It was known that the total scalar curvature of a tubular surface of small radius about a curve in a space of constant curvature also depends only on the length of the curve and the radius. As a continuation of the above research, B. Csikós and M. Horváth [2] proved that in a harmonic manifold, the surface volume, the total mean curvature, and the total scalar curvature of a tubular hypersurface of small radius about a curve also depend only on the length of the curve and the radius. They proved the following much stronger form of the above conjecture: If in a Riemannian manifold (with some natural small lower bounds on the dimension), one of the before mentioned geometric invariants of a tube of small radius about a *geodesic segment* depends only on the length of the segment and the radius of the tube, then the space is harmonic. For harmonic spaces, all four geometric invariants of tubes were computed explicitly in terms of the volume density function.

B. Csikós [3] studied the volume of Boolean expressions of large congruent balls in the n-dimensional Euclidean space. It was known that if the centers of the balls are fixed, then the volume is the sum of a Laurent series $\sum_{i=-\infty}^{n} a_i r^i$ for large r. An explicit way was found to express the coefficients a_n, a_{n-1} and a_{n-2} in terms of the system of the centers and the Boolean expression. The main coefficient a_n depends only on the Boolean expression and can take only two different values. For the union of the balls, a_{n-1} is known to be proportional to the mean width of the system of the centers. Thus, we can think of the the coefficient a_{n-1} for an arbitrary Boolean expression as a generalization of the mean width. Some known properties of the mean width were extended to these "Boolean generalizations" of the mean width.

Recently I. Gorbovickis has published a paper on some applications of the "central set" to Kneser–Poulsen type problems. With suitable modifications of the ideas of Gorbovickis, B. Csikós and M. Horváth [4] proved that if some disks in the hyperbolic, or Euclidean plane, or in a hemisphere are rearranged so that the distances between the centers of the disks do not increase, then the perimeter of the convex hull of the disks does not increase, and the area of the intersection of the disks does not decrease. These results extend earlier results of R. Alexander, V.N. Sudakov, V. Capoyleas–J. Pach, I. Gorbovickis, and K. Bezdek–R. Connelly.

B. Csikós wrote a paper [5] "On the volume of Boolean expressions of balls – A review of the Kneser Poulsen conjecture", which is essentially a review of known results around the Kneser-Poulsen conjecture, but it also contains miscellaneous results of the author that have not been published earlier.

In 2014 B. Csikós, L. Pyber and E. Szabó [6] found a counterexample to a conjecture of E. Ghys. The conjecture said that for any compact smooth manifold M, there is a constant c_M such that every finite subgroup of the diffeomorphism group of M contains an abelian subgroup of index at most c_M. Learning about the counterexample, E. Ghys modified his conjecture by replacing the word "abelian" with the word "nilpotent". It seems that B. Csikós, L. Pyber and E. Szabó have found a proof of the modified Ghys conjecture. At present the 60 page long proof is under rigorous verification and it is planned to be submitted when the authors double check its correctness.

L. Fehér and R. Rimányi [7] calculated the equivariant Chern-Schwartz-MacPherson (csm) classes of matrix Schubert varieties. These classes are generalizations of both the classical Segre classes and characteristic classes. They contain information on several enumerative geometric problems. Previously only one example was calculated. It is hoped that these classes will provide the "atoms" of the theory, analogously to the Schur polynomials in studying the Grassmannians. The technical difficulty was that these objects are formal infinite sums, so L. Fehér and R. Rimányi needed a description using generating functions and the theory of iterated residues.

L. Fehér, R. Rimányi, and A. Weber [8] generalized the csm calculations of matrix Schubert varieties to motivic Chern classes. They also proved that these classes are related (equal in some sense) to Mihalkin's K-theoretic stable envelopes.

L. Fehér and J. Nagy [9] solved Erdős-Heilbronn type problems. It turned out that equivariant cohomology, especially the Atiyah-Bott-Berline-Vergne integration formula is an efficient tool to study additive combinatorics. They showed that basic tools of the field, like the combinatorial Nullstellensatz, or the coefficient lemma are special cases of it. They reproved the Erdős-Heilbronn da Silva-Hamidoune theorem, improved results of Sun, and solved a simultaneous version of the Erdős-Heilbronn problem (the "grasshopper problem"). Using symplectic Grassmannians and flag manifolds, they gave signed versions of these theorems. E.g., given a set of integers (or mod p residue classes) of cardinality n, they give a lower bound of the numbers which are k term sums of different elements or their negatives.

Beyond the above results, substantial progress was made in the following directions, which are close to completion.

Jointly with Á. Matszangosz, L. Fehér developed a theory which is analogous to the theory of conjugacy spaces. In particular they showed that the cohomology ring of an even dimesional partial flag manifold is isomorphic to the corresponding complex flag manifold (the isomorphism halves the degree), and the Schubert calculus is also isomorphic, which can be used to study enumerative algebraic geometry over the field of reals.

A closely related result is giving an algorithm to calculate the cohomology ring of not necessarily even flag manifolds. This result was also obtained by Rabelo and San Martin in an unpublished work, but their method is essentially different.

In an old project with A. Némethi on showing that holomorphic maps of projective spaces are maximally singular, the last piece of the puzzle was obtained.

D. Szeghy [10] investigated differentiability of horizons. Let (M, g) be a Lorentz manifold. A topological hypersurface $H \subset M$ is called a horizon if it is (locally) achronal and for every point $p \in H$, there is a past inextendable, past directed, light-like geodesic γ in H. If γ is inextendable, then it is called a generator. D. Szeghy proved the following theorem.

For every generator $\gamma\colon [(\alpha,\beta) \to H$, a unique parameter $t_0 \in [\alpha,\beta]$ exists, such that there is a $k \geq 1$ for which
(1) *H is exactly of class C^k at every $\gamma(t)$, for which $t > t_0$;*
(2) *H is differentiable, but not of class C^1 at every $\gamma(t)$, for which $t \in (\alpha, t_0]$.*

He could sharpen case (1) of this theorem as follows.

If the differentiability jumping point exists on γ, i.e. $t_0 \in (\alpha,\beta)$, then $k = 1$.

Thus the differentiability can change only from C^1 to "simple" differentiability. This result can be applied to lightlike hypersurfaces, where only differentiability is assumed, since these will be locally horizons. Usually smoothness is assumed for the causal boundary of asymptotically simple spacetimes which is a lightlike hypersurface. Since the proofs use only C^1 differentiability properties of the causal boundary, we could weaken the original definition and apply the above result also in this case.

A geometric characterization of differentiability jumping points was also given:

Let $\gamma\colon [(\alpha,\beta) \to H$ be a generator and assume that the differentiability jumping point $\gamma(t_0)$ of γ exists, i.e. $t_0 \neq \alpha,\beta$. Moreover, let $s > t_0$ and $R_{\gamma(s)} \subset H$ be an $(m-2)$-dimensional space-like submanifold through $\gamma(s)$ which is at least of class C^1. Then, every point $\gamma(t)$, $t \in [\alpha, t_0]$ is an image of a non-injectivity point under the mapping $\exp|_{NR_{\gamma(s)}}$, where $NR_{\gamma(s)}$ is the normal bundle of $R_{\gamma(s)}$.

It is known that at a point $p \in H$, the horizon is differentiable if and only if p is the endpoint of only one generator. The endpoint property of the generator in this case can be described also by a geometric property:

Let $\gamma : [0,\beta) \to H$ be a generator of the globally achronal horizon H and assume that its differentiability jumping point $\gamma(t_0)$, $t_0 \in (0,\beta)$ exists, and $N(\gamma(0)) = 1$, i.e. H is differentiable at $\gamma(0)$. Moreover, let $s \in [t_0,\beta)$ and $R_{\gamma(s)} \subset H$ be an $(m-2)$-dimensional space-like submanifold through $\gamma(s)$ which is at least of class C^1. Then $\gamma(0)$ is an image of a cut vector of $R_{\gamma(s)}$, i.e. it is the cut point of γ with respect to $R_{\gamma(s)}$.

D. Szeghy also studied smooth isometric actions of a non-compact Lie group on a Lorentzian manifold. His previous results indicated an analogue of the principal orbit type theorem for infinitesimal orbit types. Though the boundary between the normalizable and non-normalizable orbits is still not fully understood to prove the conjectured analogue, the following theorem was proved in this direction:

Let $G \times M \to M$ be a smooth affine action of a Lie-group G on a smooth manifold with an connection (M, ∇). Let k be the co-dimension of the maximal dimensional orbits. Consider the union $N \subset M$ of normalizable orbits, where $N \neq \emptyset$. If the boundary of the closure of N is non-empty, then there is a k-dimesional totally geodesic submanifold through every point of this boundary.

D. Szeghy also has a joint work in progress with J. Szenthe. A space-time is spherically symmetric if there is an isometric action $\Phi\colon SO(3) \times M \to M$ on it, where the maximal dimensional orbits are of dimension 2. It is known from earlier works of J. Szenthe, that under some fairly general conditions, the union of the maximal dimensional orbits yield a dense set in M which is a warped product $L \times_\varrho \mathbb{S}^2$, where L is a leaf transversal to the orbits. Leaves can be defined as the maximal integral submanifolds of the normal distribution. A totally geodesic submanifold P is called a transverse submanifold, if any leaf that intersects P is contained in P and such a leaf is open in P. It is known that through every point of M, there is a transverse submanifold of M and any two such transverse submanifolds are isometric.

A Birkhoff field on M is a Killing vector field, which is transversal to the maximal dimensional orbits. The goal is to describe spherically symmetric spacetimes which admit a Birkhoff field. The following partial result was obtained in this direction:

Assume that M has a non-trivial Birkhoff field X and that there is a point p on a maximal dimensional orbit and the integral curve γ of X through p such that the induced action of X on γ has a fixed-point. Then every transverse submanifold is isomorphic to one of the fixed point models.

Fixed point models are a family of 2-dimensional Lorentz manifolds we can give explicitly, where the action of X on the models is also given. This result with some previous ones can describe M and the action of X on it precisely. The case when γ does not have a fixed point is under examination.

One of the main topics R. Szőke was working on is the question of uniqueness in geometric quantization, i.e. showing the independence of the quantum Hilbert space on the chosen polarization. For this to make sense one needs to look at this problem only in a reasonable family of polarizations dictated by the symmetries of the system. The family he was concerned with consists of the adapted complex structures (acs) associated to the Riemannian metric of the configuration space.

The notion of acs grew out of the investigations of certain global solutions of the complex homogeneous Monge-Ampére equation on Stein manifolds [11]. Later L. Lempert and R. Szőke [12] pointed out that the acs comes as an element in a natural family of Kähler structures, parametrized by the upper half plane and one can put this family of complex structures to form a big complex manifold, an analogue of a twistor space. Using this twistor space, L. Lempert and R. Szőke [13], applied the procedure of geometric quantization, to quantize $TM = T^*M$. This yields for each choice of a Kähler structure in the family, a Hilbert space H_s^{corr}, of L^2-holomorphic sections of a certain hermitian holomorphic line bundle. The family H^{corr} is an example, what was called in [13], a field of Hilbert spaces. These objects are more general than Hilbert bundles, but it is still possible to define smooth and real-analytic structure on such a field and talk about its curvature. As was shown by L. Lempert and R. Szőke [13], flatness (or projective flatness) implies that this field is an honest (in fact trivial) Hilbert bundle and the parallel translation naturally identifies the different fibers, i.e., quantization is unique.

In [13] it was shown that for any compact Lie group, the field H^{corr} is flat hence in this case quantization is unique. In [14] this investigation was extended further to look at the cases when the configuration space is a compact rank-1 Riemannian symmetric space and it was shown that in these cases quantization is unique only for the 3-sphere.

In this project, the methods of [14] were extended to study the cases when the configuration space is a compact Riemannian symmetric space of any rank.

This case is much harder, since the integrals expressing the curvature of the field of quantum Hilbert spaces, are multiple integrals unlike the rank 1 case. The integral depends on a positive real parameter τ. We were able to prove a version of multivariable Watson's lemma with the help of which we obtained an asymptotic expansion of the curvature integral as $\tau \to 0$. The other main ingredient was to determine the asymptotic of the curvature integral when $\tau \to \infty$. This used the methods of spherical representations and Harish-Chandra's $\mathbf{c}$ function.

Comparing the results obtained by these two asymptotic methods R. Szőke [15] could prove the following theorem: when the configuration space is a compact Riemannian symmetric space, the field H^{corr} is projectively flat iff M is a compact Lie group equipped with a biinvariant metric, i.e. quantization is unique (among compact Riemannian symmetric spaces) only for group manifolds.

Another related question we studied is that of the possible smooth structures on the field of prequantum Hilbert spaces.

R. Szőke [16] proved the following general result: let $\mathcal{A}_+$ be the group of orientation preserving invertible affine transformations of $\mathbb{R}$. Let M be a compact Riemannian manifold and T^*M its cotangent bundle equipped with the Liouville volume form. Let H be the Hilbert space of L^2 functions defined on T^*M. Then there exists a natural continuous (but not smooth) unitary representation $\rho\colon \mathcal{A}_+ \to U(H)$ of $\mathcal{A}_+$. With the help of this, the following theorem was proved: if the adapted complex structure of M is defined on the entire tangent bundle, then on the Hilbert field of prequantum Hilbert spaces, produced by geometric quantization, there exists two natural but inequivalent smooth Hilbert bundle structure.

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