Report

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1 Multivariate incomplete polynomials on starlike domains

The now classical notion of incomplete polynomials was introduced by G. G. Lorentz: for a given $0 < \theta < 1$, the polynomials of the form

$$p_n(x) = \sum_{n\theta \leq k \leq n} a_k x^k, \qquad x,\ a_k \in \mathbb{R},\ n \in \mathbb{N}$$

are called *θ-incomplete*. By a famous result due Saff and Varga [21] and Golitschek [7] a function $f \in C[-1,1]$ can be uniformly approximated by a sequence of θ-incomplete polynomials if and only if it vanishes on $[-\theta, \theta]$.

Our goal was to study multivariate θ-incomplete polynomials given by

$$P^d_{n,\theta} := \operatorname{span}\{x_1^{k_1} ... x_d^{k_d} : \theta n < k_1 + ... + k_d \leq n\}, \qquad 0 < \theta < 1.$$

As in the univariate case, one is interested in characterizing those $\mathbf{0}$-symmetric compact starlike domains $\Omega \in \mathbb{R}^d,\ d > 1$, which have the property that any $f \in C_\theta(\Omega) := \{f \in C(\Omega) : f \equiv 0 \text{ on } \theta\Omega\}$ can be approximated uniformly on Ω by θ-incomplete polynomials. It was was verified by J. Siciak [22] that θ-incomplete polynomials are dense in $C_\theta(\Omega)$ whenever Ω is a $\mathbf{0}$-symmetric *convex body*.

We introduced in [14] a new class of $\mathbf{0}$-symmetric starlike domains which includes conic sections of both convex and certain non-convex hyperbolic domains for which θ-incomplete polynomials are dense in $C_\theta(\Omega)$. In particular this family includes convex domains given earlier in [22], but it also contains essentially non-convex bodies. On the other hand we also showed that if Ω is sufficiently *concave* then the density fails. Thus the class of $\mathbf{0}$-symmetric stars Ω for which θ-incomplete polynomials are dense in $C_\theta(\Omega)$ turns out to be *essentially wider* than the family of convex bodies (or their conic sections), but this class does not include *all* $\mathbf{0}$-symmetric stars.

Furthermore extending the multivariate Bernstein polynomial method from a simplex to a general convex body we also obtained quantitative estimates for the approximation by θ-incomplete polynomials on $\mathbf{0}$-symmetric convex bodies. In fact we verified that when Ω is a $\mathbf{0}$-symmetric convex body, we can approximate all $f \in C_{\theta_1}(\Omega),\ 0 < \theta < \theta_1$ by $p_n \in P^d_{n,\theta}$ with the rate $O(\omega_2(f, n^{-1/(d+3)})_\Omega)$, where ω_2 is the second order modulus of smoothness on Ω. When Ω is a simplex (or a convex body which is an intersection of simplexes), a better rate $O(\omega_2(f, n^{-1/2})_\Omega)$ independent of d can be deduced.

2 Multivariate "needle" and fast decreasing polynomials

In this section we give an overview of our new results on a certain class of multivariate polynomials which attain value 1 at a given point and decrease exponentially as we move away from this point. These so called "needle" polynomials resemble the behavior of the Dirac delta function and they are widely used in different areas of analysis, in particular the interpolation theory, or the study of Christoffel functions. Denote by P_n^d the set of polynomials in d variables of degree at most n.

Definition 1. *Given a compact set $K \subset \mathbb{R}^d$ and $\alpha \leq 1$ we shall say that K possesses α-needle polynomials at a point $\boldsymbol{x}_0 \in K$ if for any $0 < h < 1$ and $n \in \mathbb{N}$ there exist polynomials $p \in P_n^d$ such that $0 \leq p(\boldsymbol{x}) \leq 1, \boldsymbol{x} \in K, p(\boldsymbol{x}_0) = 1$ and*

$$p(\boldsymbol{x}) \leq e^{-cnh^{\alpha}}, \quad \boldsymbol{x} \in K \setminus B(\boldsymbol{x}_0, h) \tag{1}$$

with some constant $c > 0$ depending only on K and $\boldsymbol{x}_0$.

It is well known that when $d = 1$ and K is an interval there exist $\frac{1}{2}$-needle polynomials at the endpoints and 1-needles for inner points of the interval. It can be shown that *1-needle polynomials always exist at any point* $\mathrm{x}_0 \in K$ of a compact set $K \subset \mathbb{R}^d, d \geq 1$.

A complete characterization of those points in the *convex body* for which $\frac{1}{2}$-needle polynomials exist is provided by the next theorem which can be found in [10].

Theorem 1. *Let $K \subset \mathbb{R}^d, d \geq 2$ be a convex body and $\boldsymbol{x}_0 \in \partial K$. Then K possesses $\frac{1}{2}$-needle polynomials at $\boldsymbol{x}_0$ if and only if $\boldsymbol{x}_0$ is a vertex, i.e. K possesses d linearly independent normals at $\boldsymbol{x}_0$.*

In addition, it is verified in [10] that the existence of α-needle polynomials with $\frac{1}{2} < \alpha < 1$ yields that the domain can not be of higher than $C^{2\alpha}$ smoothness in any 2-dimensional cross section.

Theorem 2. *Assume that the convex domain $K \subset \mathbb{R}^d$ possesses α-needle polynomials with $\frac{1}{2} < \alpha < 1$ at some $\boldsymbol{x}_0 \in \partial K$. Then K can not be C^{β} with $2\alpha < \beta \leq 2$ in any 2-dimensional cross section of K containing $\boldsymbol{x}_0$.*

In the past 25 years there has been an increasing interest in the study of the so called *fast decreasing polynomials* introduced by Ivanov and Totik [8] which exhibit somewhat similar properties to the needle polynomials.

As shown in [8] given a continuous increasing function ϕ on $[0, 1], \phi(0) = 0$ there exist polynomials of degree n satisfying $|p_n(x)| \leq Ce^{-cn\phi(|x|)}, \ |x| \leq 1, \ p_n(0) = 1, \ \ n \in \mathbb{N}$ if and only if

$$\int_0^1 \frac{\phi(x)}{x^2} dx < \infty. \tag{2}$$

These are the fast decreasing polynomials with respect to *inner* points.

Similarly, for a continuous increasing function ψ on $[0, 1], \psi(0) = 0$ there exist polynomials of degree n satisfying $|p_n(x)| \leq Ce^{-cn\psi(x)}, \ 0 \leq x \leq 1, \ p_n(0) = 1, \ \ n \in \mathbb{N}$ if and only if

$$\int_0^1 \frac{\psi(x)}{x^{\frac{3}{2}}} dx < \infty. \tag{3}$$

In a recent paper [11] we initiated the study of the **multivariate fast decreasing polynomials**. As can be seen above even in the univariate case there is an essential difference between the fast decreasing polynomials at the inner and boundary points. This phenomenon becomes even more intricate in the multivariate case. It is shown in [11] that the rate of decrease of the multivariate fast decreasing polynomials at the boundary of star like domains is closely related to the *smoothness of the boundary* at the corresponding points. Namely, the smoother is the boundary the slower is the rate of decrease of fast decreasing polynomials at the corresponding points. In particular, it turns out that conditions (2) and (3) characterize fast decreasing polynomials at C^2 and C^1 points of the boundary, respectively. Moreover, a new intermediate condition

$$\int_0^1 \frac{\phi(x)}{x^{1+\frac{\alpha}{2}}} dx < \infty \tag{4}$$

was shown to yield the existence of fast decreasing polynomials at $C^\alpha, 1 < \alpha < 2$ points on the boundary of the domain.

3 Christoffel functions on convex and starlike domains

The n-th Christoffel function for a measure μ is given by

$$\lambda_n(\mu, \mathbf{x}) = \frac{1}{K_n(\mu, \mathbf{x},\ \mathbf{x})}.$$

Its reciprocal admits the well known extremal property

$$K_n(\mu, \mathbf{x},\ \mathbf{x}) := D_n(w, K, \mathbf{x}) := \sup_{p \in P_n^d} \frac{p^2(\mathbf{x})}{\int_K p^2(\mathbf{y}) w(\mathbf{y}) d\mathbf{y}}. \tag{5}$$

Christoffel functions and reproducing kernels play a fundamental role in the theory of orthogonal polynomials and their study has been in the center of attention for many decades. In the univariate case the asymptotics of the Christoffel function has been studied by numerous authors. The study of the asymptotics of **multivariate** Christoffel functions poses a rather complex problem which has been solved only for some model cases: cube, unit ball, simplex.

In [12] we initiated the study the magnitude of the multivariate Christoffel functions for general starlike and convex domains. Of course getting precise asymptotics for Christoffel functions is not a visible task in such generality. Nevertheless, we were able to obtain sharp pointwise upper bounds which reveal the behavior of Christoffel functions near the boundary of the domain, and their dependence on the geometry of the domain.

Let us formulate one of our main results in this direction.

Theorem 3. *Let $K \subset \mathbb{R}^d$ be a $C^\alpha, 0 < \alpha \leq 2$ domain which is starlike about the origin. Consider the measure $d\mu = wd\boldsymbol{x}$, where w is continuous and positive in* $\mathrm{Int}K$. *Then there exists a constant $A > 0$ depending only on K and d such that*

$$\limsup_{n\to\infty} n^{-d} D_n(w, K, \mathbf{x}) \leq \frac{A}{w(\mathbf{x})(1 - \varphi_K(\mathbf{x}))^{\gamma(\alpha,d)}}, \quad \mathbf{x} \in \mathrm{Int}K, \tag{6}$$

where

$$\gamma(\alpha, d) := \frac{1}{2} + \frac{(d-1)(2-\alpha)}{2\alpha}.$$

and $\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \frac{\mathbf{x}}{\alpha} \in K\}$ is the Minkowski functional of the domain.

One can easily observe that when $\alpha = 1$ we have $\gamma(1, d) = \frac{d}{2}$ for domains with Lip1 boundary, which is consistent with the known estimate for the simplex or cube. Moreover, if $\alpha = 2$ we get $\gamma(2, d) = \frac{1}{2}$ for C^2 domains which is consistent with the well known estimate for the ball. It turns out that $\gamma(\alpha, d)$ is in general, the correct exponent in the range $1 < \alpha < 2$, as well. In [12] this was shown to be the case up to a $\log n$ factor. Subsequently in a recent paper Prymak [20] even this extra $\log n$ factor was removed.

4 Marcinkiewicz-Zygmund type inequalities and optimal meshes on multivariate domains

The classical Marcinkiewicz-Zygmund inequality states that for any univariate trigonometric polynomial of degree at most n and $1 \le p < \infty$

$$\int |T_n|^p \sim \frac{1}{n} \sum_{s=0}^{2n} \left| T_n\left(\frac{2\pi s}{2n+1}\right)\right|^p \tag{7}$$

where the constants involved depend only on p. This inequality is a basic tool for the discretization of the L^p norms of trigonometric polynomials. In the past 30 years Marcinkiewicz-Zygmund type inequalities for trigonometric and algebraic polynomials with various weights were widely used in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation. In univariate case a far reaching generalization of (7) for the so called *doubling weights* was given by Mastroianni and Totik [18].

Typically a Marcinkiewicz-Zygmund type result on a general compact set K consists in finding a discrete point sets $Y_N = \{y_1, ..., y_N\} \subset K$ of cardinality $N \sim n^d$, and proper positive numbers $a_j > 0, 1 \le j \le N$, $\sum_{1\le j\le N} a_j \sim 1$ so that for every $g \in P_n^d$ we have

$$\|g\|^p_{L^p(w)} \sim \sum_{1\le j\le N} a_j |g(y_j)|^p, \tag{8}$$

where $\|g\|_{L^p(w)}$ stands for the weighted L^p norm with a weight w. We will call discrete point sets $Y_N = \{y_1, ..., y_N\} \subset K$ with the above property MZ meshes.

The requirement that the cardinality of the discrete set Y_N satisfies $N \sim n^d$ leads to an asymptotically smallest possible discrete mesh, because $\dim P_n^d \sim n^d$ and (8) can not hold with fewer points than the dimension of P_n^d.

Recently, certain Marcinkiewicz-Zygmund type results for doubling weights on the sphere and ball were given by Feng Dai [6] using the somewhat technical notion of maximal separability. In a recent paper [5] we established simple explicit MZ meshes which do not require the technical condition of maximal separability. In addition we extended the above Marcinkiewicz-Zygmund type results to much more general multivariate domains, which in particular include polytopes, cones, spherical sectors, tori, etc. Our approach relied on application of various polynomial inequalities including Bernstein-Markov, Schur and Videnskii type estimates, and also on using symmetry and rotation to generate results on new domains.

The above notion of MZ meshes is closely related to the notion of **admissible meshes** or norming sets introduced in [4]. Admissible meshes $Y_N \subset K$ have the property

$$\max_{x\in K} |g(x)| \sim \max_{x\in Y_N} |g(x)|, \quad \forall g \in P_n^d.$$

Evidently, the MZ meshes can be considered as the L^p analogues of the optimal meshes. If in addition, $N \sim n^d$ then the admissible mesh is called **optimal**. It has been conjectured some time ago (see [9]) *that every convex body in $\mathbb{R}^d$ possesses an optimal mesh.* In [9] a special case was resolved: it was shown that star like C^2-domains and convex polytopes in $\mathbb{R}^d$ possess optimal meshes. In an upcoming paper [13] we derived some new multivariate tangential Markov type inequalities and used them in order to provide a complete positive answer to the above conjecture for $d = 2$.

Theorem 4. ([13]) *Every convex body in $\mathbb{R}^2$ possesses an optimal mesh.*

5 Bernstein-Markov type inequalities for asymmetric weights

Let $K \subset \mathbb{R}$ be any compact set and $\|p\|_K := \sup_{x\in K} |p(x)|$ the usual supremum norm on K. The classical Bernstein problem consists in estimating the derivative of the polynomial $p'(x)$ for a given $p \in P_n$, $\|p\|_K = 1$ and $x \in \text{Int } K$. Typically, this estimate is given in terms of the degree n of the polynomials and the distance of point $x \in \text{Int } K$ to the boundary ∂K of the compact K. This problem goes back to Bernstein and Markov who showed respectively, that

$$\|\sqrt{(x-a)(b-x)}p_n'(x)\|_{[a,b]} \le n\|p_n\|_{[a,b]} \tag{9}$$

and

$$\|p_n'\|_{[a,b]} \le \frac{2n^2}{b-a}\|p_n\|_{[a,b]}, \quad p_n \in P_n \tag{10}$$

with both inequalities being sharp.

Various extensions of the Bernstein and Markov type inequalities for more general domains and norms have been widely investigated in the past decades. In [18] the case of rather general *weighted* uniform norms on the interval was studied. Let $A^\star$ denote the set of integrable weights $w(x) \ge 0$ ($x \in [a,b]$) satisfying the inequality

$$\|w\|_E \le \frac{C}{|E|}\int_E w(t)\,dt \tag{11}$$

for every interval $E \subset [a,b]$ with some constant $C \ge 1$. Then as shown in [18] for any $w \in A^\star$ and $p_n \in P_n$

$$\|\sqrt{(x-a)(b-x)}wp_n'\|_{[a,b]} \le cn\|wp_n\|_{[a,b]}, \quad \|wp_n'\|_{[a,b]} \le cn^2\|wp_n\|_{[a,b]}, \tag{12}$$

where the constants above depend only on w, a, b. Thus the n-th order Bernstein and n^2 order Markov upper bounds extend for $A^\star$ weights, as well.

The condition $A^\star$ imposed on the weights is rather general, in particular it includes all Jacobi type weights $\prod_j |x-x_j|^{\gamma_j}$ which allow the weight to vanish as a power of x. In a very recent paper (12) was extended to a wider class of weights which may vanish exponentially. However, all of the above classes of weights require that the weight has certain *symmetry*, that is it vanishes to the left and to the right of the given point with equal speed. We initiated in [16] the study of the Bernstein–Markov type inequalities for the so called *asymmetric weights* which may vanish at a given point with different rates. A typical asymmetric weight is given by the following function which has a power discrepancy at the origin:

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha, & \text{if} \quad x \le 0, \\ x^\beta, & \text{if} \quad x > 0, \end{cases} \qquad 0 \le \alpha \le \beta\,. \tag{13}$$

This weight does not belong to $A^\star$ if $\alpha < \beta$.

We gave in [16] some new Bernstein type inequalities for such asymmetric Jacobi type weights. In contrast to the estimates provided previously for the symmetric weights in the asymmetric case the resulting bounds for the derivatives of n-th degree polynomials are typically of order n^γ, $\gamma > 1$. We also provided certain converse estimates showing that the increase of the rate of derivatives in the asymmetric case is in general unavoidable. In particular, even a logarithmic asymmetry of the weight may cause the Bernstein factor to be of order greater than $O(n)$. The converse estimates for the asymmetric weights rely heavily on the needle and fast decreasing polynomials discussed in Section 2.

6 Polynomial and rational operators

(A) In [23] we considered the modified Lagrange interpolatory operator introduced by Bernstein. Compared to Lagrange interpolation, these operators interpolate at less points, but they converge for all continuous functions.

For an $f \in C[-1,1]$ consider the Lagrange interpolatory operator

$$L_n(f,x) := \sum_{i=1}^{n} f(x_i)\ell_i(x),$$

where

$$x_i := x_{in} = \cos t_i, \quad t_i := \frac{2i-1}{2n}\pi, \qquad i = 1, 2, \ldots, n \tag{14}$$

are the roots of the Chebyshev polynomial $T_n(x) := \cos nt,\ x = \cos t$, and $\ell_i(x)$ are the fundamental polynomials of Lagrange interpolation. It is well-known that Lagrange interpolation cannot be uniformly convergent for all continuous functions, no matter what are the nodes of interpolation. In order to achieve better convergence properties, S. N. Bernstein introduced the following modification of Lagrange interpolation. Let l be a fixed positive integer, $n \ge 2l$, and consider the linear operator

$$\begin{aligned} B_{n,l}(f,x) :=& \sum_{i=1}^{m}\sum_{k=1}^{2l-1} f(x_{2(i-1)l+k})\left\{\ell_{2(i-1)l+k}(x) + (-1)^{k+1}\ell_{2il}(x)\right\} \\ &+ \sum_{k=1}^{n-2lm} f(x_{2lm+k})\ell_{2lm+k}(x), \qquad m := \left[\frac{n}{2l}\right], \end{aligned} \tag{15}$$

where the last sum (call it the "tail part") contains at most $2l-1$ terms, and when n is a multiple of $2l$ it does not appear at all. In fact, this is a slightly modified form of the original operator of Bernstein (established by L. Szili and P. Vértesi). This operator represents a polynomial of degree at most $n-1$, reproduces constants, and has the interpolatory property

$$B_{n,l}(f, x_{2(i-1)l+k}) = f(x_{2(i-1)l+k}), \quad i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, 2l-1$$

and

$$B_{n,l}(f,x_{2lm+k}) = f(x_{2lm+k}), \quad k=1,2,\ldots,n-2lm\,.$$

This means that the interpolation holds at

$$m(2l-1)+n-2lm = n-m \geq n\left(1-\frac{1}{2l}\right)$$

points, i.e. by increasing l we increase the number of interpolations.

The main idea in this definition is that the sum or difference of the two fundamental functions of interpolation in the formula (15), in most subintervals determined by adjacent nodes, is always smaller than the fundamental functions themselves. This is how Bernstein was able to prove that

$$\lim_{n\to\infty}\|f-B_{n,l}(f)\| = 0 \tag{16}$$

for all $f\in C[-1,1]$.

Our main goal in [23] was to give direct and converse error estimates for the above modified interpolatory operator in order to quantify Bernstein's original results .

(B) Let $f(x)\in C[-1,1]$, $-1=x_{0n}<x_{1n}<\cdots<x_{nn}=1$ a set of nodes, and consider the linear operator

$$B_n(f,x) := \frac{\sum\limits_{k=0}^{n}\dfrac{(-1)^k}{x-x_{kn}}f(x_{kn})}{\sum\limits_{k=0}^{n}\dfrac{(-1)^k}{x-x_{kn}}}\,. \tag{17}$$

This is a rational function of degree at most n, called *barycentric interpolation*, since it interpolates the function at $n+1$ points:

$$B_n(f,x_{kn}) = f(x_{kn}), \qquad k=0,\ldots,n\,.$$

While B_n is not a positive operator, its degree is just 1 less than the number of interpolation points. We gave in [19] sharp upper bounds for the rate of approximation of the operator B_n. In fact we proved that for every continuous functions

$$\|f(x)-B_n(f,x)\| \leq c\omega\left(f,\frac{1}{n}\right)\log n \tag{18}$$

with an absolute constant $c>1$, where $\omega(f,\cdot)$ is the modulus of continuity of f and $\|\cdot\|$ is the supremum norm over the interval $[-1,1]$. Here the log-factor, in general, cannot be dropped.

7 Approximation on several intervals

(A) We considered in [17] Lagrange interpolation on a union of finitely many disjoint intervals and obtained lower and upper estimates for the corresponding Lebesgue constant. In fact, we proved that the classic Faber theorem which states that for any system of nodes the order of Lebesgue constants is at least $c\log n$, remains true in this setting as well. This looks like a trivial consequence of the original Faber theorem, but in fact we need a deep result of Erdős and Vértesi [3] which says that not only the Lebesgue constant, but also the Lebesgue function is at least $c\log n$ on a "large" set. We also constructed systems of nodes which show that the mentioned lower estimate can be attained. The special case of two intervals of equal lengths is simpler, and an explicit construction for two non-symmetric intervals was also given.

(B) A polynomial $p_m \in \Pi_m$ is called the *Chebyshev polynomial on* J_s if $\|p_m\|_{J_s} = 1$ and its leading coefficient is maximal among all polynomials of degree at most m having norm 1 on J_s. This polynomial is known to be unique. Clearly, $J_s \subset p_m^{-1}([-1,1])$ where

$$p_m^{-1}(K) := \{x \in \mathbb{R} : p_m(x) \in K\}$$

denotes the inverse polynomial image of the set $K \subset \mathbb{R}$. The characteristic property of the Chebyshev polynomial p_m is the fact that it attains values $1, -1$ with alternating signs at $m+1$ distinct consecutive points in J_s.

We say that the Chebyshev polynomial p_m is the *T-polynomial on* J_s if $|p_m(x)| = 1$ at $m+s$ points of J_s. (This notion was introduced by F. Peherstorfer.) Whenever p_m is the T-polynomial on J_s it follows that $J_s = p_m^{-1}([-1,1])$ and for each $|y| < 1$ the equation $p_m(x) = y$ has exactly m solutions inside J_s. In addition, it is also known that p_m has $m-s$ extremal points *inside* J_s where $|p_m(x)| = 1$ and all $2s$ boundary points of J_s are also extremal. Clearly, this yields that $s \le m$.

While Chebyshev polynomials exist for every compact set in $\mathbb{R}$, the existence of T-polynomials of given degree m depends on the structure of J_s and m. For example, in the simplest case $s = m = 2$, T-polynomial exists only if the pair of intervals is symmetric. When $s = 2, m = 3$, the set $J_2 = [-1,a] \cup [b,1]$ possesses a T-polynomial if and only if $a + b = 1 - \frac{(b-a)^2}{4}$. There is also an alternative way of describing those systems of disjoint intervals which possess T-polynomials. One can choose an arbitrary polynomial $p_m \in \Pi_m$ with m distinct real zeros and such that all its local extremal values are ≥ 1 in absolute value. Then setting $J_s := p_m^{-1}([-1,1])$ we shall obtain a system of $s(\le m)$ disjoint intervals for which (after rescaling to $[-1,1]$), p_m will be the corresponding T-polynomial.

One should also note that even though not all systems of disjoint intervals possess T-polynomials it was shown by Peherstorfer and Totik that any system of intervals can be approximated up to any degree of precision by systems of intervals having T-polynomials. We explored in [15] this fact in order to show that for any system of disjoint intervals there exist convergent interpolatory processes of order $(1+\epsilon)n$ based on n points. This extends an earlier result by Erdős, Kroó and Szabados from a single interval to several intervals.

We also provided an estimate for the Lebesgue function of any system of nodes derived by the inverse polynomial image in the presence of T-polynomials. It led to a nice estimate for the Lebesgue constant in case when the system of nodes is *admissible* in the sense that

$$\frac{\|\omega_n\|_{[-1,1]}}{|\omega_n(\pm 1)|} = O(1), \quad n \in \mathbb{N}. \tag{19}$$

The admissibility condition (19) essentially means that at the endpoints ± 1 the fundamental polynomial $\omega_n(y)$ attains values of order $\|\omega_n\|_{[-1,1]}$. We examined the necessity of condition (19) by showing that the Lebesgue constant goes to infinity as the smallest and largest nodes tend to the end points of $[-1,1]$.

Summarizing, we can see that the inverse polynomial images of systems of nodes may preserve the order of the Lebesgue functions and Lebesgue constants when the systems of intervals J_s possess T-polynomials. This leads to the following natural question: is the T-property necessary for the inverse polynomial image to work? We gave an affirmative answer to this question by showing that without the T condition the order of the Lebesgue constants for the inverse polynomial images of **all** systems of nodes is substantially larger than the optimal order $\log n$.

It is plausible that without the T condition the order of the Lebesgue constants for the inverse polynomial images of all systems of nodes have exponential increase, i.e., the lower estimate can be essentially improved. We verified that this is indeed the case for two disjoint intervals, that is when $s = 2$.

Even though the norm of Lagrange operator of degree n is of order $\log n$, it is well known that the situation changes dramatically when the strict condition on the number of nodes of interpolation is loosened. Namely, for any $\varepsilon > 0$ with a proper choice of nodes in $[-1, 1]$ one can construct polynomials of degree n which interpolate at $[n(1-\varepsilon)]$ points and approximate all $f \in C[-1, 1]$ with the optimal order of best approximation.

Erdős, Kroó and Szabados gave a complete characterization of systems of nodes with the above properties on the interval $[-1, 1]$. It turns out that the inverse polynomial image method allows to construct such interpolating processes on *any systems of disjoint intervals*, as well.

(C) Assume that J_s has a T-polynomial $p(x) \in \Pi_m,\ m \geq s$, normalized such that $p(0) = 0$. For $n \in \mathbb{N}$, let $x_{k1} < \cdots < x_{km_k}$ be defined by

$$p(x_{ki}) = \frac{k}{n}, \qquad i = 1, \ldots, m_k; \quad k = 0, \ldots, n$$

where

$$m_k = \begin{cases} m + s - \left[\frac{m+s}{2}\right], & \text{if} \quad k = 0, \\ m, & \text{if} \quad k = 1, \ldots, n-1, \\ \left[\frac{m+s}{2}\right], & \text{if} \quad k = n. \end{cases}$$

The existence of such x_{ki}'s follows from the properties of T-polynomials.

For an arbitrary $f(x) \in C(J_s)$, let

$$L_k(f, x) = \sum_{i=1}^{m_k} f(x_{ki}) \ell_{ki}(x) \in \Pi_{m_k - 1}, \qquad k = 0, 1, \ldots, n\,,$$

be the Lagrange interpolation polynomial with respect to the nodes x_{ki}. Here $\ell_{ki}(x) \in \Pi_{m_k-1}$ are the fundamental polynomials.

We considered the discrete linear operator

$$B_n(f, x) := \sum_{k=0}^{n} L_k(f, x) b_{nk}(p(x)), \qquad x \in J_s\,, \tag{20}$$

where

$$b_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, \ldots, n\,,$$

are the fundamental functions of the Bernstein polynomials. Evidently, $B_n(f, x) \in \Pi_{mn+m-1}$, and there are $mn + s$ function values used in the construction of the operator. This means that the difference between the number of function values and the degree of the operator is $m - s + 1$, i.e. independent of n, just like in case of the classic Bernstein polynomials.

Although this is not a positive operator, it still has a bounded norm. We proved a pointwise convergence estimate for this operator (see [24]).

References

[1] S. M. Alsulami, P. Nevai, J. Szabados and W. van Assche, *A family of nonlinear difference equations: Existence, uniqueness, and asymptotic behavior of positive solutions*, J. Approx. Theory, **193** (2015), 39-55.

[2] B. Della Vecchia, G. Mastroianni and J. Szabados, *A weighted generalization of Szász–Mirakyan and Butzer operators*, Mediterranean J. Math., **12** (2015), 433-454.

[3] P. Erdős and P. Vértesi, On the Lebesgue function of interpolation, in: *Functional Analysis and Approximation*, eds. P. L. Butzer, B. Sz.-Nagy, E. G"orlich, ISNM vol. 60, Birkhäuser (1981), pp. 299-309.

[4] J.P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory, **152** (2008), 82-100.

[5] S. De Marchi and A. Kroó, *MarcinkiewiczZygmund type results in multivariate domains,* Acta Math. Hungar., **154**(2018), 69-89.

[6] Feng Dai, *Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights*, J. Funct. Anal. **235** (2006), 137-170.

[7] M. v. Golitschek, *Approximation by incomplete polynomials*, J. Approx. Theory **28** (1980), 155-160.

[8] K. Ivanov and V. Totik, *Fast decreasing polynomials*, Constr. Approx., **6** (1990), 1-20.

[9] A. Kroó, *On optimal polynomial meshes*, J. Approx. Theory, **163** (2011), 1107-1124.

[10] A. Kroó, *Multivariate "needle" polynomials with application to norming sets and cubature Formulas*, Acta Math. Hungar., **147** (2015), 46-72.

[11] A. Kroó, *Multivariate fast decreasing polynomials,* Acta Math. Hungar., **148** (2016), 101-119.

[12] A. Kroó, *Christoffel functions on convex and star like domains in $\mathbb{R}^d$*, J. Math. Anal. Appl., **421** (2015), 718-729.

[13] A. Kroó, *On the existence of optimal meshes in every convex domain on the plane*, J. Approx. Theory, (2018), to appear.

[14] A. Kroó and J. Szabados, *On multivariate incomplete polynomials on star like domains*, Construct. Approx., **39** (2014), 397-419.

[15] A. Kroó and J. Szabados, *Inverse polynomial mappings and interpolation on several intervals*, J. Math. Anal. Appl., **436** (2016), 1165-1179.

[16] A. Kroó and J. Szabados, *Polynomial inequalities with asymmetric weights*, Jaen J. Approximation (accepted).

[17] A. Lukashov and J. Szabados, *The order of Lebesgue constant of Lagrange interpolation on several intervals*, Periodica Math. Hungar., **72** (2016), 103-111.

[18] G. Mastroianni and V. Totik, *Weighted polynomial inequalities with doubling and A_∞ weights*, Constr. Approx. **16**(2000), 37-71.

[19] G. Mastroianni and J. Szabados, *Barycentric interpolation on equidistant nodes*, Jaen J. Approximation, **9** (2017), 25-36.

[20] A. Prymak, *Upper estimates of Christoffel function on convex domains,* J. Math. Anal. Appl. **455** (2017), 1984-2000.

[21] E. B. Saff and R. S. Varga, *The sharpness of Lorentz's theorem on incomplete polynomials,* Trans. Amer. Math. Soc. **249** (1979), 163-186.

[22] J. Siciak, On approximation by incomplete multivariate polynomials, in: *Complex Analysis and Digital Geometry,* Proc. Conference on the occasion of Christer Kiselman's retirement (Uppsala, Sweden, 2006), ed. M. Passare (2009), pp. 311-321.

[23] J. Szabados, *On a quasi-interpolating Bernstein operator*, J. Approx. Theory, **196** (2015), 1-12.

[24] J. Szabados, Bernstein polynomials on several intervals, in: *Progress in Approximation Theory and Applicable Complex Analysis - In the memory of Q. I. Rahman*, Springer Optimization and its Applications, Volume 117, Springer Verlag, (2017), pp. 301-316.