Closing report of the research project K-111697
“Group-theoretic aspects of integrable systems and their dualities”

The main goal of the project has been the investigation of classical, finite-dimensional integrable Hamiltonian system and their duality relations. We studied integrable systems of point particles moving in one dimension: members of the celebrated family of Calogero–Moser–Sutherland type models, and their relativistic deformations given by Ruijsenaars–Schneider–van Diejen models. We also investigated such extensions of these systems by internal ‘spin’ degrees of freedom, which arise as the gauge invariant content of low dimensional Yang–Mills and Chern–Simons field theories. Among these is the self-dual, compactified trigonometric Ruijsenaars–Schneider model that we studied at the quantum mechanical level, too. We first summarize our results on these topics, then sketch further results. Below, we refer to the 14 publications supported by the project, which are freely available in the arXiv (https://arxiv.org/), too.

I. Dualities of integrable many-body systems [1,3,4,8,9,14]. Among the most fascinating features of integrable many-body systems are the *duality relations* that convert the particle positions of one system into the action variables of another one, and vice versa. This intriguing phenomenon, originally discovered by Ruijsenaars, is often called action-angle duality since the related systems live on each other’s action-angle phase spaces. The group theoretic approach to action-angle dualities aims at treating all dual pairs as particular cases of the following scenario. One starts with an unreduced master phase space that carries *two* commuting families of ‘free’ Hamiltonians, and performs reduction relying on a symmetry under which both sets of Hamiltonians are invariant. The key step is to develop two models of the reduced phase space, in such a way that the two families of free Hamiltonians descend to action and position variables of many-body systems in terms of both models, but their role is interchanged in the two models. Any two models of the reduced phase space are related automatically by a canonical transformation, and this canonical transformation gives the duality relation between the two many-body systems obtained by a single Hamiltonian reduction. We have treated two infinite series of dual pairs in this approach. Namely, first we treated in [1] the duality between the trigonometric BC(n) Sutherland model, governed by the Hamiltonian

$$H_{\mathrm{Suth}}(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+\sum_{1\le j<k\le n}\left(\frac{\gamma}{\sin^2(q_j-q_k)}+\frac{\gamma}{\sin^2(q_j+q_k)}\right)+\sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)}+\sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)},$$

and the special real form of the complex rational Ruijsenaars–Schneider–van Diejen model *locally* given by the Hamiltonian

$$\begin{aligned}H_{\mathrm{RSvD}}(\lambda,\theta)=&-\frac{\nu\kappa}{4\mu^2}\prod_{j=1}^{n}\left(1-\frac{4\mu^2}{\lambda_j^2}\right)+\frac{\nu\kappa}{4\mu^2}\\&+\sum_{j=1}^{n}\cos(\theta_j)\left(1-\frac{\nu^2}{\lambda_j^2}\right)^{\frac{1}{2}}\left(1-\frac{\kappa^2}{\lambda_j^2}\right)^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left(1-\frac{4\mu^2}{(\lambda_j-\lambda_k)^2}\right)^{\frac{1}{2}}\left(1-\frac{4\mu^2}{(\lambda_j+\lambda_k)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Here, q_i, p_j and λ_i, θ_j are Darboux coordinates, and duality turns out to hold under the following relation between the coupling constants: $\gamma = \mu^2$, $2\gamma_1 = \nu\kappa$ and $2\gamma_2 = (\nu - \kappa)^2$. We have derived this dual pair by reduction of two natural free systems on the cotangent bundle $T^*\mathrm{U}(2n)$. It is very important that the above formula of H_{RSvD} is actually valid only on a dense open subset of the full reduced phase space, and the major achievement of the paper [1] was the clarification of the global structure of the second model of the reduced phase space, on which the Hamiltonians flow of H_{RSvD} is complete. We have also used the duality to draw conclusions about the qualitative nature of the dynamics of these integrable systems.

In the follow up paper [4], we clarified the connection between two complete sets of of commuting Hamiltonians that underlie the Liouville integrability of an analytic continuation of the above rational RSvD Hamiltonian. The Hamiltonian in question, in which one has positive signs under the square roots and $\cosh(\theta_j)$ instead of $\cos(\theta_j)$, is maximally superintegrable, i.e., it admits $(2n-1)$ globally smooth constants of motion. Therefore there are several possibilities for selecting n constants of motion, containing the Hamiltonian itself, that form an Abelian Poisson algebra of maximal rank, as required for Liouville integrability. One distinguished choice arises by taking the classical limit of the commuting difference operators, which were obtained by van Diejen in his study of the quantum mechanical version of these models. Another choice is provided by taking the coefficient of the characteristic polynomial of the Lax matrix of the model, which was recently found by Pusztai. We have shown that these two choices are equivalent, i.e., they functionally generate the same commutative algebra of constants of motion.

One of the main achievements of the project is the complete analysis of the 'relativistic' deformation of the dual pair represented by the above Hamiltonians, which is published in the paper [14]. The deformed Hamiltonians contain a parameter, akin to the velocity of light, and reproduce the previously studied system in a limit. The term 'Poisson-Lie deformation' in the title of [14] refers to the fact that, instead of ordinary symmetry, a Poisson–Lie symmetry was utilized for the construction of the dual pair by Hamiltonian reduction. Poisson–Lie symmetries represent the classical limit of quantum group symmetries. In a nutshell, Poisson–Lie symmetry means that the symmetry group G is itself endowed with a Poisson bracket, $\{\cdot,\cdot\}_G$, which is compatible with the group multiplication and is such that the symmetry action $G \times M \to M$, where $(M, \{\ ,\ \}_M)$ is the unreduced phase space, is given by a Poisson map. In our case $(M, \{\ ,\ \}_M)$ was taken to be the so-called Heisenberg double of the standard Poisson–Lie group $\mathrm{SU}(2n)$, which is a natural generalization of the cotangent bundle of $\mathrm{SU}(2n)$, introduced in the pioneering work of Semenov-Tian-Shansky on Poisson–Lie symmetries. As a manifold, this Heisenberg double is given by $\mathrm{SL}(2n, \mathbb{C})$, regarded as a real Lie group. It carries two natural Abelian Poisson algebras of free Hamiltonians, constructed out of invariant functions associated with the two factors in the Iwasawa decomposition of the elements of $\mathrm{SL}(2n, \mathbb{C})$. Our construction was based on a symmetry group of the form $G = G_L \times G_R$, where $G_L = G_R = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$ is the block-diagonal Poisson-Lie subgroup of $\mathrm{SU}(2n)$. In [14], we have analyzed in detail two global models of the reduced phase space, revealing non-trivial features of the two systems in duality with one another. We established that the symplectic vector space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ underlies both global models, and showed that for both systems the action variables generate the standard torus action on $\mathbb{C}^n$, and the fixed point of this action corresponds to the unique equilibrium positions of the pertinent systems. The systems in duality were proved to be non-degenerate in the sense that the functional dimension of the Poisson algebra of their conserved quantities is equal to half the dimension of the phase space. Our systems represent the first examples of dual pairs having the property that the joint level surfaces of the action variables are compact for both members of the dual pair. As a side result, we also pointed out a formula for the eigenvalues of the quantized Hamiltonians, which results from a semi-classical, 'oscillator quantization' of the action variables. A comparison of this formula with the result of the 'Shchrödinger quantization', which in this case gives difference operators for the quantum Hamiltonians, is still an open problem for future work.

The work that culminated in [14] took a long time, and partial results of the analysis were published in [3] and [8]. Concretely, in [3] we described one of the members of the dual pair, paying attention to the global structure of the associated model of the reduced phase space. The findings of this paper further developed those obtained earlier by Marshall on an analytic continuation of the system. In the article [8] we worked out the description of the dual systems focusing on a dense open submanifold of the reduced phase space. As is often the case, such a dense open submanifold is specified by imposing the condition that the eigenvalues of a some matrix are all distinct. This leads to technical simplifications, but finally one should either justify the non-degeneracy of those eigenvalues, or should also explore the part of the phase space where non-degeneracy does not hold. In fact, we devoted a separate paper [9] to exploring the full phase space of the analytically continued system studied by Marshall adopting restriction to an open submanifold.

II. Constructions of many-body models with spin related to field theories [2,13]. In the paper [2], we presented generalizations of certain spin Sutherland type systems obtained earlier by Blom and Langmann and by Polychronakos in two different ways: from SU(n) Yang–Mills theory on the cylinder and by constraining geodesic motion on the N-fold direct product of $\mathrm{SU}(n)$ with itself, for any $N > 1$. Our models are in correspondence with the Dynkin diagram automorphisms of arbitrary connected and simply connected compact simple Lie groups. We gave a finite-dimensional as well as an infinite-dimensional derivation and shed light on the mechanism whereby they lead to the same classical integrable systems. The infinite-dimensional approach, based on Yang–Mills fields with twisted boundary conditions (alias twisted current algebras) was inspired by an analogous derivation of the spinless Sutherland model due to Gorsky and Nekrasov. The finite-dimensional method relies on Hamiltonian reduction under twisted conjugations of N-fold direct product groups, linking the quantum mechanics of the reduced systems to representation theory.

In the paper [13], we constructed new Poisson–Lie analogues of the standard spin Sutherland models containing 'collective' spin variables in addition to the positions and momenta of point particles moving on the circle. The standard spin Sutherland Hamiltonians are given in Lie-theoretic terms as follows:

$$\mathcal{H}_{\mathrm{spin-Suth}}(q,p,\xi) = \frac{1}{2}\langle p,p\rangle + \frac{1}{2}\sum_{\alpha>0}\frac{1}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2\frac{\alpha(q)}{2}}.$$

Here, $\langle\ ,\ \rangle$ is the Killing form of the complexification of the Lie algebra $\mathcal{G}$ of a compact simple Lie group, G, $e^{\mathrm{i}q}$ belongs to the regular part in the maximal torus $\mathbb{T} < G$, $\mathrm{i}p$ varies in the Lie algebra $\mathcal{T}$ of $\mathbb{T}$, and α runs over the corresponding positive roots. The spin variable $\xi = \sum_{\alpha>0}(\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha})$ lies in the intersection of an arbitrarily chosen coadjoint orbit of G and the annihilator subspace of $\mathcal{T}$. These models correspond to Hamiltonian reductions of 'free motion' on cotangent bundles of compact simple Lie groups based on the conjugation action. Our new models were obtained by reducing the corresponding Heisenberg doubles with the aid of a Poisson-Lie analogue of the conjugation action. We described the reduced symplectic structure and demonstrated that the 'reduced main Hamiltonians' reproduce the spin Sutherland model by keeping only their leading terms. We also explained how the solutions of the equations of motion emerge from geodesics on the compact Lie group via the standard projection method and exhibited many first integrals. Our new models can be regarded as special real forms of complex holomorphic models that provide an effective description of the moduli space of flat $G^{\mathbb{C}}$ connections on a torus with a puncture. Alternatively speaking, this moduli space is the phase space of complex Chern-Simons field theory on the torus, such that the holonomy of the field around the puncture is fixed to some conjugacy class in the pertinent complex simple Lie group $G^{\mathbb{C}}$. These holomorphic integrable systems were recently studied by Reshetikhin.

III. Investigation of self-dual Ruijsenaars–Schneider models [5,11]. The trigonometric Ruijsenaars–Schneider model admits several different real forms. We studied the self-dual real form given by the local Hamiltonian

$$H(q,p) \equiv \sum_{j=1}^{n} \cos(p_j) \prod_{k\neq j}^{n} \left[1 - \frac{\sin^2 y}{\sin^2 \frac{(q_j - q_k)}{2}}\right]^{\frac{1}{2}},$$

where the variables $\delta_j = e^{\mathrm{i}q_j}$ $(j = 1, ..., n)$ are interpreted as the positions of n 'particles' moving on the circle and the canonically conjugate momenta p_j encode the compact variables $\Theta_j = e^{-\mathrm{i}p_j}$, subject to the center of mass condition $\prod_{j=1}^{n} \delta_j = \prod_{j=1}^{n} \Theta_j = 1$. In a classical paper from 1995, Ruijsenaars realized that the model admits complete Hamiltonian flows if the coupling constant y is restricted to the range $0 < y < 2\pi/n$ and the local phase space is compactified on the complex projective space $\mathbb{CP}^{n-1}$. In a paper by Fehér and Kluck from 2014, the range of admissible parameters y was greatly extended. Moreover, it was shown that there are two qualitatively different compactifications, called type (i) and type (ii), the former having the global phase space $\mathbb{CP}^{n-1}$. This was found via constructing the compactified models by the technique of quasi-Hamiltonian reduction, which is a powerful approach for describing phase spaces of Chern-Simons field theories on Riemann surfaces. Both the type (i) and type (ii) trigonometric systems are self-dual in the sense of action-angle duality. In the framework of this project [5], we first provided a direct approach to the type (i) models. This turned out to work also for the elliptic interaction 'potential' based on the Weierstrass $\wp$-function. Then we quantized the type (i) instances of the trigonometric models [11]. We explicitly solved the joint eigenvalue problem for the corresponding quantum Hamiltonians by generalizing previous results of van Diejen and Vinet, who studied the traditional $0 < y < 2\pi/n$ case. The quantum Hamiltonians were defined as discrete difference operators acting in a finite-dimensional Hilbert space of complex-valued functions supported on a uniform lattice over the classical configuration space, and their joint eigenfunctions were found in terms of discretized Macdonald polynomials.

The results described up to this point correspond to realization of items in the research plan of the project. However, as often happens, we could not fully realize the plan. In particular, we could not perform the investigation of deformations and discretizations of WZNW models, which was envisioned as part of the tasks for the second half of the project. By that time the interest in this question diminished, and exciting new problems stemmed from the performed research. In my opinion, this does not represent a significant departure from the research plan, since plans in theoretical research are usually tentative and flexible.

IV. Miscellaneous further results [6,7,12]. Realizing new ideas that arose in the course of the work, the following further results were obtained. In the publication [6], we presented a new and very simple proof of Sklyanin's formula for spectral Darboux coordinates of the rational Calogero–Moser system. In particular, we used Hamiltonian reduction to simplify Falqui and Mencattini's recent proof of Sklyanin's conjectured expression for the canonical conjugates of the eigenvalues of the Lax matrix of the system. In the paper [7], a Lax pair for the classical hyperbolic van Diejen system with two independent coupling parameters was discovered, and also a proposal was mode concerning generalization of the result to a 3-parametric case. The Lax pair was used for setting up an algorithm for obtaining the solutions of the equations of motion. These results represented a significant step towards finding Lax pairs for the most general, five parameter family of trigonometric/hyperbolic van Diejen systems. (In the meantime, the general problem was solved by O. Chalykh.) Finally, in [12] we investigated the Hamiltonian structure of a finite dimensional dynamical system derived by Braden and Hone in 1996 from the solitons of A_{n-1} affine Toda field theory. This system of evolution equations was interpreted originally as a limiting special case of the spin Ruijsenaars–Schneider models due to Krichever and Zabrodin. We noticed that, by utilizing a change of variables, the same dynamics can be interpreted also as a special case of the spin Sutherland systems obtained by reducing the free geodesic motion on the symmetric space of positive Hermitian matrices. We then proved that the alternative interpretations lead to two different Hamiltonian descriptions of the model, such that the two Poisson brackets are compatible. Thus we uncovered a bi-Hamiltonian structure for the Braden–Hone system. It appears that similar bi-Hamiltonian structures exist for a host of related models as well, and this issue is currently under investigation.

V. Training of young researchers. Based on results obtained as part of the project, the junior participant, T.F. Görbe, defended his PhD thesis [14] at the University of Szeged. The outstanding quality of his work was acknowledged by the awarding of the prestigious *Pro laudanda promotione* prize of the university. Then he successfully applied for a Marie Curie research fellowship, and from September 2018 works at one of the important scientific centers of our subject, at the University of Leeds. In addition to supervising this PhD thesis, the project leader also supervised a BSc and an MSc thesis devoted to problems related to the project, which represent further educational side results.

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