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Brief description of the results

The project supported and coordinated the research of 20 participants. Since the beginning of 2015, the research group has published 147 scientific papers in international journals and also 1 textbook, 1 monograph, 1 PhD thesis and 2 habilitation dissertations. We organized several international conferences:

1. The 16^{th} Debrecen–Katowice Winter Seminar in 2016 in Hernádvécse;
2. The 54^{th} International Symposium on Functional Equations in 2016 in Hajdúszoboszló;
3. The Conference on Inequalities and Applications 2016 in Hajdúszoboszló;
4. The 12^{th} International Symposium on Generalized Convexity and Monotonicity in 2017 in Hajdúszoboszló;
5. The 18^{th} Debrecen–Katowice Winter Seminar in 2018 in Hajdúszoboszló;
6. Spring Workshop on Analysis in 2018 in Debrecen;
7. The conference Numbers, Functions and Equations 2018 in Hajdúszoboszló;
8. The Second Spring Seminar on Analysis in 2019 in Debrecen.

Solution methods for functional equations

We proved characterization theorems for generalized exponential polynomials in (Székelyhidi, [131]). The aim of (Gselmann, [56]) was to characterize the determinant function on the set of positive definite $n \times n$ matrices with entries from a subfield $\mathbb{F}$ of the reals. In (Lajkó–Mészáros, [89]) we solved some special cases of the pexiderized Hosszú equation on the intervals $[0, 1]$ and $(0, 1)$.

In (Gselmann [57]), the general and also the continuous solutions of the discrete wave equation were determined. In (Székelyhidi–Wilkens, [151]), we proved that spectral synthesis holds for a variety on an Abelian group if spectral analysis holds on it and the residue ring of the annihilator of the variety is a Noether ring. This extends the fundamental result of M. Lefranc.

The results of (Maksa–Sablik, [97]) are about the alienation of the exponential Cauchy and the Hosszú functional equations. Provided that h is continuous, the equation $g(x+y) - g(x)g(y) = h(x+y-xy) + h(xy) - h(x) - h(y)$ can hold on the interval $(0, 1)$ if and only if the left and right hand sides are identically zero separately. In (Fechner–Székelyhidi, [42]), a generalization of the sine-cosine equation over commutative topological groups was solved completely. In (Fechner–Székelyhidi, [43]), sine functions on different types of hypergroups, including polynomial, Sturm–Liouville hypergroups and some non-commutative hypergroup were also introduced and investigated.

In (Bessenyei–Konkoly–Szabó, [17]), we studied single-variable linear functional equations that involve one unknown function and a finite set of known functions forming a group under composition. The main results provided the complete description of the solution set. In (Fechner–Székelyhidi, [44]) the complex-valued solutions defined on a double coset hypergroup of the exponential, additive, and quadratic functional equations were described. Moreover, the m-sine functions on a double coset hypergroup were discussed. In (Fechner–Székelyhidi, [45]) we dealt with trigonometric functional equations on hypergroups. We described the general continuous solution of sine and cosine addition formulas and a so-called sine-cosine functional equation on a locally compact hypergroup. In (Székelyhidi–Vati, [150]), we studied functional equations on hypergroup joins. These equations characterize basic function classes, like exponential, additive, quadratic, and sine functions.

In (Maksa, [95]), we gave the solution of a problem formulated in Kominek and Sikorska in connection with the strong alienation of the logarithmic and the exponential Cauchy equations. In (Mészáros, [102]) we considered a multiplicative type functional equation derived from the pexiderized Davison equation on different structures.

Regular and irregular solutions of functional equations

For functional equations there are standard theorems proving that all measurable solutions can be replaced uniquely by continuous solutions. In some cases, for example, for multiplicative equations, these methods can be applied only if we also prove that the solutions are nonzero, too. In (Járai, [81]), we considered such an example and proved also a general theorem. A 0-1 law for generalized 'multiplicative type' functional equations over manifolds was verified in measure theoretical and also in category theory settings in (Járai, [82]).

In (Székelyhidi, [137]), a simple proof is given showing that the graph of an additive function is either connected or totally disconnected. This was extended in (Almira–Boros, [3]), where it was proved that the graph of a discontinuous n-monomial real function is either connected or totally disconnected.

In (Kiss–Páles, [88]), a functional equation related to the equality problem of two-variable weighted quasi-arithmetic means is solved under minimal regularity assumptions.

Characterization problems via functional equations

The aim of (Gselmann–Páles, [64]), was to characterize the additive solvability and the linear independence of the solutions of a system of functional equations which is related to higher-order derivations. In (Gselmann, [55]) we provided sufficient conditions for an additive function to be a real derivation: assuming that for a differentiable function f and an additive function a, the function $a \circ f - f' \cdot a$ is regular (e.g. measurable, continuous, locally bounded), then a is the sum of a derivation and a linear function. The purpose of (Grünwald–Páles, [54]) was to show that functions that differentiate the two-variable product function and one of the exponential, trigonometric or hyperbolic functions are derivations.

In (Boros–Fechner, [25]), we proved that if a generalized polynomial function f satisfies the condition $f(x)f(y) = 0$ for all solutions of the equation $x^2 + y^2 = 1$, then f is identically zero. Furthermore, discontinuous monomial functions with connected graph are characterized as those satisfying a certain big graph property. In (Boros–Fechner–Kutas, [26]), we showed that real additive or quadratic functions f such that $(x, y) \mapsto f(x)f(y)$ is locally bounded on a the unit circle, are continuous. In (Boros–Garda-Mátyás, [27]), quadratic functions f satisfying $y^2 f(x) = x^2 f(y)$ for $(x, y) \in \mathbb{R}^2$ that fulfill the condition $P(x, y) = 0$ for some fixed polynomial P of two variables, were considered. For special polynomials, it was proved that in this case f has to be continuous.

Local polynomials on Abelian groups were characterized by a local Fréchet-type functional equation in (Almira–Székelyhidi, [4]). The results were applied to generalize Montel's Theorem and to obtain Montel-type theorems on commutative groups. In (Almira–Székelyhidi, [5]), some classes of local polynomial functions on Abelian groups were characterized by the properties of their variety. It is also shown that a generalized polynomial is a polynomial if and only if its variety contains finitely many linearly independent additive functions. In (Almira–Székelyhidi, [8]), assuming that a linear space of real polynomials in d variables is given which is translation and dilation invariant, it is proved that if a sequence in this space converges pointwise to a polynomial, then the limit polynomial also belongs to the space. In (Almira–Székelyhidi, [7]), local exponential monomials and polynomials on different types of Abelian groups were characterized and Montel–type theorems were established.

In (Daróczy–Maksa, [39]) the connection between the measurable solutions of a famous identity, the Abel functional equation and the dilogarithm functions is established. In (Glavosits–Lajkó, [53]) three new logarithmic type functional equations were introduced and their general solutions were completely determined. Two generalizations of the cocycle equation have been introduced, investigated and compared to former variants in (Száz, [117]).

In (Gselmann–Kiss–Vincze, [62]) some multivariate and univariate characterizations of higher order derivations were proved. This method allowed to refine the process of computing the solutions of univariate functional equations of the form $\sum_{k=1}^{n} x^{p_k} f_k(x^{q_k}) = 0$, where the unknown functions $f_1, \ldots, f_n \colon R \to R$ are additive on the ring R. The aim of (Gselmann–Kiss–Vincze, [63]) was to prove if $n \in \mathbb{N}$, $\mathbb{K}$ a field and $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{C}$ are additive functions such that $\sum_{i=1}^{n} f_i^{q_i}(x^{p_i}) = 0$ holds on $\mathbb{K}$, then the functions $f_1, \ldots, f_n$ are linear combinations of field homomorphisms from $\mathbb{K}$ to $\mathbb{C}$. The five-chapter survey (Gselmann, [60]) was about characterization problems of derivations via functional equations. Here we showed (among others) that: derivations can be characterized by one single functional equation; the additive solvability of a system of functional equations was also investigated (as a consequence of the main result, for any nonzero real derivation d, the iterates $d^0, d^1, \ldots, d^n$ of d were shown to be linearly independent, and the graph of the mapping $x \mapsto (x, d^1(x), \ldots, d^n(x))$ to be dense in $\mathbb{R}^{n+1}$); knowing the action of an additive mapping on a given elementary function, under which additional conditions can we deduce that this mapping is a derivation or a linear function.

Spectral analysis and spectral synthesis

In (Székelyhidi, [129]) it was shown that spectral synthesis holds for a variety, if the factor ring with respect to its annihilator in the group algebra is a Noetherian semi-local ring with exponential maximal ideals. In (Székelyhidi, [132]) the class of exponential monomials on non-discrete locally compact Abelian groups was investigated using an extended form of the annihilator method. The main result of (Székelyhidi, [135]) stated that, for any discrete Abelian group, spectral synthesis holds for every variety whose annihilator ideal is finitely generated. In (Székelyhidi, [134]), the powers of maximal ideals in the measure algebra of some locally compact Abelian groups in terms of the derivatives of the Fourier–Laplace transform of compactly supported measures were described. It was shown that if the locally compact Abelian group has sufficiently many real characters, then all derivatives of the Fourier–Laplace transform of a measure at some point of its spectrum completely characterize the measure. Furthermore, the derivatives of the Fourier–Laplace transform of a measure can be used to describe the powers of the maximal ideals corresponding to the points of the spectrum of the measure on discrete Abelian groups with finite torsion-free rank. As an application of (Székelyhidi, [141]), the extension of L. Schwartz's fundamental result was established. Due to counterexamples of D. I. Gurevich, there is no straightforward extension to higher dimensions. The new idea was to replace translations by proper Euclidean motions in higher dimensions. For this purpose 'translation invariance' was replaced by invariance with respect to a compact group of automorphisms. The role of exponential functions was then be played by spherical functions. In (Székelyhidi, [144]) spherical monomials on some types of affine groups using invariant differential operators were described. In particular, it was also shown that if the algebra of invariant differential operators is generated by a single operator then the classes of spherical monomials and of spherical moment functions coincide. In (Székelyhidi–Tabatabaie, [147]) the recent developments, results and problems of spectral analysis, spectral synthesis and their applications was treated. In 2004, a counterexample was given for a 1965 result of R. J. Elliott claiming that discrete spectral synthesis holds on every Abelian group. Characterizations of the Abelian groups that possess spectral analysis and spectral synthesis, respectively, were published in 2005. A characterization of the varieties on discrete Abelian groups enjoying spectral synthesis is still missing. In (Székelyhidi–Wilkens, [152]) a ring theoretical approach to this issue was presented and a generalization of the Principal Ideal Theorem on discrete Abelian groups was also proved.

The aim of (Fechner–Székelyhidi, [46]) was to characterize generalized moment functions on a non-commutative affine group. For a locally compact group G and its compact subgroup K, the connection between K-spherical functions on G and exponentials on the double coset hypergroup $G//K$ was presented. They also gave the general description of generalized moment functions on Aff K and specific examples for $K = \mathrm{SO}(n)$, and on the so-called $ax + b$-group. Spherical spectral synthesis on the affine group of $SU(n)$ was established in (Székelyhidi, [146]).

Stability of functional equations

In (Székelyhidi, [130]), we obtained stability theorems for functional equations on hypergroups. In (Székelyhidi, [138]), we showed how some fundamental functional equations can be treated on certain hypergroups. Stability and superstability results using invariant means and other tools were also presented. In (Székelyhidi, [139]) invariant means on and amenability of double coset spaces was studied and amenability of Gelfand pairs was proved. In (Székelyhidi, [142]) stability-type theorems for functional equations related to spherical functions were proved.

A form of the asymptotic stability of the Cauchy and Jensen functional equations with exact stability constants was proved in (Bahyrycz–Páles–Piszczek, [9]). In (Gselmann–Kelemen, [61]) results demonstrating the stability behaviour for ordinary delay differential equations have been established.

Equality, comparison and characterization of means, invariance equations

In (Matkowski–Páles, [101]), we characterized the so-called generalized quasi-arithmetic means that were introduced by Matkowski in 2010. The characterization involves the Gauss composition of the cyclic mean-type mapping induced by the generalized quasi-arithmetic mean and a generalized bisymmetry equation. In (Kiss–Páles, [86]), we introduced the notion of the descendant of a sequence of means. We proved that the descendant of a sequence of weighted quasi-arithmetic is also a weighted quasi-arithmetic mean. More general statements were obtained for Matkowski means. It was proved that if a function f is convex w.r.t. a sequence of means then it is also convex w.r.t. all descendants.

A functional equation involving pairs of means were considered in (Daróczy–Totik, [40]). It was shown that there are only constant solutions if continuous differentiability is assumed, and there may be non-constant everywhere differentiable solutions. Various other situations are considered, where less smoothness is assumed on the unknown function. In (Daróczy–Jarczyk–Jarczyk, [38], elaborating an idea of the construction of means, the notion of marginal joints of means is introduced.

The aim of (Páles–Pasteczka, [107]) was to characterize in broad classes of means the so-called Hardy means, i.e., those means M that satisfy the inequality $x_1 + M(x_1, x_2) + \cdots + M(x_1, ..., x_n) + \cdots \leq C(x_1 + x_2 + \cdots + x_n + \cdots)$ for all positive sequences (x_n) with some finite positive constant C. The main results offers a characterization of Hardy means in the class of symmetric, increasing, Jensen concave and repetition invariant means and also a formula for the sharpest constant C. In (Páles–Pasteczka, [109]) we determined the sharp Hardy constant in the cases when the mean M is either a concave quasi-arithmetic or a concave and homogeneous deviation mean. The main goal of (Páles–Pasteczka, [108]) was to prove the weighted counterpart of the results of [107]. In (Páles–Pasteczka, [110]) weighted Hardy type inequalities were established in the case when M is monotone and satisfies the weighted counterpart of the Kedlaya inequality. In particular, if M is symmetric, Jensen-concave, and the weight sequence satisfies a monotonicity condition. In addition, if M is a symmetric and monotone mean, then the biggest possible weighted Hardy constant is achieved if the weight sequence is constant. To explore the hidden homogeneity property in Hardy type inequalities, various notions for the homogenizations of means were introduced and investigated in (Páles–Pasteczka, [111]). Developing an extension theorem for conditionally additive functions, in (Burai, [34]), we investigated the equality problem of quasi-arithmetic expressions.

In (Kiss–Páles, [87]) a new class of means was introduced which generalizes the well-known means for arbitrary linear spaces and enjoy a so-called reducibility property. The main results gave a sufficient condition for the reducibility of the (M, N)-convexity property of functions and also for Hölder–Minkowski type inequalities. In (Páles–Zakaria, [114]), we introduced a new class of generalized Bajraktarević means and established necessary and sufficient conditions for their the local and global comparison problem. In (Burai–Jarczyk, [35]), we characterized the symmetry property in the class of so-called Makó–Páles means.

Generalizations and stability of convexity, applications

In (Gilányi–Merentes–Nikodem–Páles, [49]), we proved decomposition and a characterization theorems for strongly Wright-convex functions of higher order. In (Gilányi–Merentes–Nikodem–Páles, [50]), we investigated $(t_1, ..., t_n)$-Wright convex functions and obtained a characterization theorem via generalized derivatives. In (Gilányi–Gonzalez–Nikodem–Páles, [48]) approximate and strong convexity properties for set-valued mappings were introduced and Bernstein-Doetsch type theorems with Tabor type error terms were established in this general framework. The m-convexity of functions and sets and also their m-convex hulls was animated in (Gilányi–Merentes–Quintero, [51, 52]).

In (Páles–Radácsi, [113]) various notions of convexity of real functions with respect to Chebyshev systems defined over arbitrary subsets of the real line were introduced. The main results offered various characterizations in terms of the corresponding lower Dinghas type derivative. In (Páles–Radácsi, [112]), using a determinant identity of Sylvester, we established a formula for the generalized divided differences and obtained a new characterization of convexity with respect to Chebyshev systems. Thus, we got a necessary condition for functions which can be written as the difference of two functions which are convex with respect to a given Chebyshev system. In (Páles, [105]) a general Cauchy-type mean value theorem for the ratio of functional determinants is offered.

Regular pairs of functions induce convex structures both in an axiomatic and in an algebraic way. The purpose of (Bessenyei, [13]) was to link these structures, by showing that they coincide. In (Bessenyei–Pénzes, [21]), we extended the results of Hopf and Popoviciu to the setting of higher-order monotonicity induced by quasipolynomial Chebyshev systems. In (Bessenyei–Konkoly–Popovics, [16]), applying Beckenbach families, the notion of (planar) convexity was extended and then generalized convex functions were studied. We proved the analogue of the Radon, Helly, Carathéodory and Minkowski Theorems. In (Bessenyei–Popovics, [19]), we investigated convex structures induced by Chebyshev systems and completely described their combinatorial invariants. In (Bessenyei–Popovics, [18]) separation theorems were obtained in terms of convexity structures induced by Beckenbach families. The aim of (Bessenyei, [14]) was to characterize those pairs of real functions that possess an affine separator. In (Bessenyei–Pénzes, [22]), we showed that nonconstant h-affine functions appear only in the classical case. Affine and convex separation problems were studied without further restrictions on h.

In (Maksa–Páles, [96]) we investigated continuity properties of functions that satisfy the inequality $f(H_p(x, y)) \le H_p(f(x), f(y))$, where H_p is the pth power mean. We showed that there exist discontinuous multiplicative functions that are p-Jensen convex for all positive rational p. On the other hand, if f is p-Jensen convex for all $p \in P$, where P is a set of positive Lebesgue measure, then f must be continuous.

In (Jarczyk–Páles [80]), two parallel notions of convexity of sets were introduced in the Abelian semigroup setting. The algebraic and set-theoretic properties were investigated. A formula for the computation of the convex hull and a Stone-type separation theorem for disjoint convex sets was established. In (Boros–Nagy, [28]), we proved that approximately nth-order convex functions are nth-order convex provided that the error function has a certain asymptotic property. In (Burai, [33]), two generalized convexity notions, their properties and their use in optimization theory was investigated. In particular, we deduced first-order necessary and sufficient conditions of optimality. In (Burai–Makó, [36]), a characterization and lower Hermite–Hadamard type inequalities were obtained for certain classes of Schur-convex functions. In (Makó–Házy, [99]) a connection between strong Jensen convexity and strong convexity was established. The optimal Takagi type error-function was also determined.

The paper (Losonczi, [90]) extended the discrete Wirtinger type inequalities to the weighted case in four different ways and determined the best constants in the lower and upper estimations.

In (Boros–Nagy, [29]), (α, F)-convex functions were characterized by comparison of modified difference ratios and support properties. If α satisfies some additional conditions, then the differentiability of (α, F)-convex functions in an appropriate sense was also established. In (Boros–Száz, [31]), we obtained generalized Schwarz inequalities by introducing an appropriate notion of generalized semi-inner products on groupoids. In (Makó, [98]), we obtained a new and significantly simpler proof of the approximate convexity of the so-called Takagi function. In (Makó–Házy, [100]), we established approximate Hermite–Hadamard type inequalities for approximately convex functions.

In (Olbryś–Páles, [103]), we established a general framework in which the verification of support theorems for generalized convex functions acting between an algebraic structure and an ordered algebraic structure is still possible. By taking several particular cases, we deduce support and extension theorems in various classical and important settings.

Finsler spaces

In (Kertész–Tamássy, [85]) we considered distance spaces over $\mathbb{R}^n$, whose distance functions are differentiable. These spaces are situated between general metric spaces (distance spaces) and Finsler spaces. We investigated those curves of differentiable distance spaces, which possess the same properties as geodesics do in Finsler spaces. The characterizations of these curves were obtained without using calculus of variations but applying direct geometric considerations. In (Deng–Kertész–Yan, [41]), we proved that there are no proper Berwald–Einstein manifolds.

The space of continuous, $\mathrm{SL}(m,\mathbb{C})$-equivariant, $m \geq 2$, and translation covariant valuations taking values in the space of real symmetric tensors on $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ of rank $r \geq 0$ is completely described in (Abardia-Evéquoz–Böröczky–DomokosKertész, [1]). The classification involves the moment tensor valuation for $r \geq 1$ and is analogous to the known classification of the corresponding tensor valuations that are $\mathrm{SL}(m,\mathbb{R})$-equivariant, although the method of the proof cannot be adapted.

Walsh–Fourier series

Given a lacunary sequence of natural numbers, the a.e. convergence of the corresponding means of the two-variable Vilenkin–Fourier series for integrable functions was obtained in (Gát, [65]). The survey paper (Gát, [66]) discussed and compared several recent convergence and divergence results related to two-dimensional Walsh–Fourier series. In (Gát–Goginava, [69]), we proved that the maximal operators of the dyadic triangular-Fejér means of two-dimensional Walsh–Fourier series are of weak type $(1,1)$ and that the dyadic triangular-Fejér means of integrable functions converge a.e. In (Gát–Karagulyan, [77]) sequences of compact bounded linear operators of the $L^p(0,1)$ space with certain convergence properties were considered. Divergence type theorems for multiple sequences of tensor products of these operators were proved. These theorems imply that $L\log^{d-1}L$ is the optimal Orlicz space guaranteeing almost everywhere summability of rectangular partial sums of multiple Fourier series related to general orthogonal systems. In (Gát–Goginava, [70]), we studied the approximation by rectangular partial sums of double Fourier series on unbounded Vilenkin groups in the spaces C and L_1. Criteria of the uniform convergence and L-convergence of double Vilenkin–Fourier series was obtained.

The main purpose of (Gát–Goginava, [74]) was to prove that if $1 \leq p < 2$, then the set of functions from $L_p(\mathbb{I}^2)$ such that the subsequence of triangular partial means $S^{\Delta}_{2^A}(f)$ of the double Walsh–Fourier series is convergent in measure on $\mathbb{I}^2$, is of first Baire category in $L_p(\mathbb{I}^2)$. We also proved that, for each function $f \in \mathbb{L}_2(\mathbb{I}^2)$, a.e. convergence of $S_{a(n)}(f) \to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

In (Gát–Goginava, [75]) the boundedness of maximal operators of subsequences of (C,α_n)-means of partial sums of Walsh–Fourier series from the Hardy space H_p into the space L_p was studied. In (Gát–Lucskai, [78]) the authors demonstrated the difference of the trigonometric and the Walsh system with respect to the behaviour of the maximal function of the Fejér kernels. Moreover, properties (positivity among others) of the Walsh logarithmic kernels were also investigated. The main aim of (Gát–Lucskai, [79]) was to prove that the non-negativity of the Riesz's logarithmic kernels with respect to the Walsh–Kaczmarz system fails to hold.

Further miscellaneous results

Varoius characterizations and applications of relator spaces were considered in (Száz–Zakaria, [128]), (Száz, [116, 125, 126]).

The extensions of the Banach Fixed Point Theorem for linear quasicontractions by Ćirić and for nonlinear contractions by Matkowski have been unified in (Bessenyei [11]). In several extensions of the classical Banach Fixed Point Theorem the usual contractivity property is replaced by weaker but still effective assumptions. In (Bessenyei, [12]) simple an elementary proof for some known fixed point results is presented. In (Bessenyei–Páles, [20]), a contraction principle was developed for generalized Matkowski type contractions in semimetric spaces where the so-called triangle function of the underlying semimetric enjoys natural regularity properties. In (Lovas–Mező, [94]), the Furstenberg topology of integers was investigated.

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Publications between January 2015 and June 2019

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