Final report on OTKA project K109789

In this final report we have only included publications on which the financial support of OTKA project K109789 is acknowledged. (Other publications of the participants were either written before the start of this project, or they are not related to the topics of the project, and therefore not listed.) In the list below we give a short description of each paper, and use the numbering given in the list of publications in the online version of final report at the OTKA homepage. Most of the papers have already been published, while some of the recent ones are submitted for publication (in the latter case we have given 2019 as the year of prospective publication).

(1) We proved earlier that a relatively general even function on the real line satisfying a vanishing condition can be expanded in terms of a certain function series closely related to the Wilson functions. In this paper we show that this vanishing condition was necessary by proving that every term of the function series satisfies this condition. In fact we prove a more general identity which implies these vanishing properties.

(2) This survey type paper discusses the periodic decomposition problem from its classical, univariate real periodic case to generalizations to operators or shift transforms and discusses various connections between available results and open problems.

(3) One classical result of Freiman gives the optimal lower bound for the cardinality of $A+A$ if A is a d-dimensional finite set in R^d. Matolcsi and Ruzsa have recently generalized this lower bound to $|A+kB|$ if B is d-dimensional, and A is contained in the convex hull of B. We characterize the equality case of the Matolcsi-Ruzsa bound. The argument is based partially on understanding triangulations of polytopes.

(4) A variation on the sum-product problem seeks to show that a set which is defined by additive and multiplicative operations will always be large. In this paper, we prove new results of this type.

(5) Let $d > 2$, $A \subset \mathbb{Z}^d$ be finite and not contained in a translate of any hyperplane, and $q \in \mathbb{Z}$ such that $|q| > 1$. We show that $|A + q \cdot A| \geq (|q| + d + 1)|A| - O_{q,d}(1)$.

(6) For finite sets A and B in the plane, we write $A + B$ to denote the set of sums of the elements of A and B. In addition, we write $tr(A)$ to denote the common number of triangles in any triangulation of the convex hull of A using the points of A as vertices. We consider the conjecture that $tr(A + B)^{1/2} \geq tr(A)^{1/2} + tr(B)^{1/2}$. If true, this conjecture would be a discrete, two-dimensional analogue to the Brunn-Minkowski inequality. We prove the conjecture in three special cases.

(7) The optimal condition of the cone volume measure of a pair of antopodal points is proved and analyzed.

(8) We describe an approach to the circulant Hadamard conjecture based on Walsh-Fourier analysis. We show that the existence of a circulant Hadamard matrix of order n is equivalent to the existence of a non-trivial solution of a certain homogenous linear system of equations. Based on this system, a possible way of proving the conjecture is proposed.

(9) The linear programming (LP) bound of Delsarte can be applied to several problems in various branches of mathematics. We describe a general Fourier analytic method to get a slight improvement on this bound. We then apply our method to the problem of mutually unbiased bases (MUBs).

(10) Following previous works of Chung we are interested in Vinogradov-type inequalities for some multivariate character sums. Using Johnsen's bound on a complete unidimensional character sum we obtain a sharper result in several ranges of parameters.

(11) In this note we investigate the existence of flat orthogonal matrices, i.e. real orthogonal matrices with all entries having absolute value close to $1/\sqrt{n}$.

(12) The century old extremal problem, solved by Caratheodory and Fejer, concerns a nonnegative trigonometric polynomial normalized by $a(0) = 1$, and the quantity to be maximized is the coefficient $a(1)$. We study an analogous problem in locally compact Abelian groups.

(13) We solve unconditionally the class number one problem for the 2-parameter family of real quadratic fields $\mathbb{Q}(\sqrt{d})$ with square-free discriminant $d = (an)^2 + 4a$ for positive odd integers a and n.

(14) Recall that van der Waerden's theorem states that any finite coloring of the naturals has arbitrarily long monochromatic arithmetic sequences. We explore questions about the set of differences of those sequences.

(15) We show that the cone-volume measure of a convex body with centroid at the origin satisfies the subspace concentration condition. This extends former results obtained in the discrete as well as in the symmetric case and implies, among others, aconjectured best possible inequality for the U-functional of a convex body.

(16) Existence of solution of the logarithmic Minkowski problem is proved for the case where the discrete measures on the unit sphere satisfy the subspace concentration condition with respect to some special proper subspaces. In order to understand how optimal this condition is, we discuss certain conditions that any cone volume measure satisfies.

(17) We study connections between the cardinality of the sets $A + A$ and $A - A$ where A is a finite set in a commutative group and one of them is near to the maximal possible value. We improve earlier results of the second author

(18) For a convex polyhedron standing with one of its face on a fixed plane we mean rolling when it is rotated into another similar position around any of its edge lying on the plane. A set is said to be the trace of the polyhedron P if some point of it coincides of some vertex of P in some position. In this note we investigate the trace of deterministic and random rolling of polyhedra.

(19) We give bounds for additive and multiplicative character sums of multiplicative and additive Hilbert cubes in prime fields.

(20) A theorem of Folner asserts that for any set $A \subset \mathbb{Z}$ of positive upper density there is a Bohr neigbourhood B of 0 such that $B \setminus (A-A)$ has zero density. We use this result to deduce some consequences about the structure of difference sets of sets of integers having a positive upper density.

(21) In this paper we study properties of the maximum values of additive representation functions corresponding to nonnegative sets of integers.

(22) We investigate lower bounds of the inverse Markov factor $M_{n,q}(K)$ for certain convex domains K in the complex plane.

(23) We improve the upper bound on the density of measurable planar sets avoiding the unit distance. In higher dimensions n we show that a 1-avoiding set of block structure necessarily has density less than $1/2^n$.

(24) For a sequence $A \subset \mathbb{N}$ let $P(A)$ be the set of all sums of distinct terms taken from A. The sequence A is said to be complete if $P(A)$ contains all sufficiently large integers. We prove two new results about complete sequences.

(25) We consider a certain definite integral involving the product of two classical hypergeometric functions having complicated arguments. We show in this paper the surprising fact that this integral does not depend on the parameters of the hypergeometric functions.

(26) We study two extremal quantities corresponding to non-negative (not identically zero) positive definite functions.

(27) We give bounds on the number of distinct differences $N_a - a$ as a varies over all elements of a given finite set A, and N_a is a nearest neighbour to a.

(28) We prove several expanders with exponent strictly greater than 2.

(29) We prove that any finite set of real numbers can be split into two parts, one part being highly non-additive and the other highly non-multiplicative.

(30) We prove a minor result connected to the celebrated conjecture of Erdos and Turan that the additive representation function of a set of nonnegative integers cannot be bounded (if it is non-zero for all sufficiently large integers).

(31) We classify sets of nonnegative integers A, B such that the h-fold representation functions of A and B coincide for all large integers.

(32) We generalize a previous result of Diaconis, Shao and Soundararajan for prime-square modulus to a modulus of arbitrary square.

(33) Let A, B be sets of positive integers such that $A+B$ contains all but finitely many positive integers. Sarkozy and Szemeredi proved that if $A(x)B(x)/x \to 1$, then $A(x)B(x) - x \to \infty$. Chen and Fang considerably improved Sarkozy and Szemeredi's bound. We further improve their estimate and show by an example that our result is nearly best possible.

(34) A stability version of the reverse isoperimetric inequality, and the corresponding inequality for isotropic measures are established.

(35) The planar L_p Minkowski problem is solved for $0 < p < 1$.

(36) We show that the cone-volume measure of a convex body with centroid at the origin satisfies the subspace concentration condition. This implies, among others, a conjectured best possible inequality for the U-functional of a convex body. For both results we provide stronger versions in the sense of stability inequalities.

(37) We investigate lower bounds of the inverse Markov factor $M_{n,q}(K)$ for a general class of convex domains K in the complex plane.

(38) We investigate lower bounds of the inverse Markov factor $M_{n,q}(K)$ for convex domains with positive depth.

(39) We characterize the situation when a linear combination of representation functions corresponding to a set A is multiplicative.

(40) In this paper we focus on partitions of the natural numbers into two sets affording identical representation functions. We solve a recent problem of Lev and Chen.

(41) Let $\Gamma \subset PSL(2,\mathbb{R})$ be a finite volume Fuchsian group. The hyperbolic circle problem is the estimation of the number of elements of the Γ-orbit of z in a hyperbolic circle around w of radius R where z and w are given points in the upper half plane and R is a large number. We improve an earlier result of Risager and Petridis regarding the error term in a special case of this problem.

(42) Ambrus and Ball conjectured that for any n translates of a (fixed) concave kernel function on the torus with only one singularity at 0, the minimal value of the maximum of the sum of translates is achieved when the translation points form a regular n-gon on the circle. Subsequently, the conjecture was proved by Hardin, Kendall and Saff. As a far-reaching genralization, we prove that considering n different translates, a proper generalization holds: the minimal value of the maximum of the sum of translates function is achieved exactly when the sum equioscillates.

(43) It is well-known that the real 0.23571113... is irrational. The earlier proofs used Dirichlet theorem and the Bertrand's postulate. In an earlier work the second named author gave a simple elementary proof of this statement (in fact a little bit more). In the present paper we revisited this theorem giving a new simple proof, which used only Gelfand's approach

(44) The celebrated conjecture of Erdos and Turan states that the additive representation function of a set of nonnegative integers cannot be bounded (if it is non-zero for all sufficiently large integers). We improve an earlier result of Li and Chen concerning this conjecture.

(45) We prove that a good average order on the Goldbach generating function implies that the real parts of the non-trivial zeros of the Riemann zeta function are strictly less than 1. This together with existing results establishes an equivalence between such asymptotics and the Riemann Hypothesis.

(46) We prove that the regular octahedron has the minimal surface area among 3-polytopes of given volume and having at most six vertices.

(47) We present a new approach to the problem of mutually unbiased bases (MUBs), based on positive definite functions on the unitary group.

(48) We apply an improvement of the Delsarte LP-bound to give a new proof of the non-existence of finite projective planes of order 6, and uniqueness of finite projective planes of order 7. The proof is computer aided, and it is also feasible to apply to higher orders like 8, 9 and, with further improvements, possibly 10 and 12.

(49) Recently, the authors of the present work (together with M. N. Kolountzakis) introduced a new version of the non-commutative Delsarte scheme and applied it to the problem of mutually unbiased bases. Here we use this method to investigate the existence of a finite projective plane of a given order d. In particular, a short new proof is obtained for the nonexistence of a projective plane of order 6. For higher orders like 10 and 12, the method is non decisive but could turn out to give important supplementary information.

(50) In the last decade the growth in groups focused the interest of many researchers in the domain of additive combinatorics. In this note, we investigate the special case of the Heisenberg group H of order 3 on different fields R, namely, $R = \mathbb{R}$ and $R = \mathbb{F}_p$.

(51) In this paper we investigate how small the density of a multiplicative basis of order h can be in $\{1, 2, ..., n\}$ and in $\mathbb{Z}_+$. Furthermore, a related problem of Erdos is also studied: How dense can a set of integers be, if none of them divides the product of h others?

(52) The century old extremal problem, solved by Caratheodory and Fejer, concerns a nonnegative trigonometric polynomial normalized by $a(0) = 1$, and the quantity to be maximized is the coefficient $a(1)$. We study a reformulation of the problem in locally compact non-Abelian groups.

(53) In this paper, we prove some results about various generalizations of the Stanley sequence.

(54) In this paper we solve an open problem of Chen and Zhou and prove several related results about a conjecture of Erdos.

(55) We establish a number of uncertainty inequalities for the additive group of a finite affine plane, showing that for p prime, a nonzero function $f : \mathbb{F}_p^2 \to \mathbb{C}$ and its Fourier transform cannot have small supports simultaneously.

(56) The arithmetic Kakeya conjecture, formulated by Katz and Tao in 2002, is a statement about addition of finite sets. It is known to imply a form of the Kakeya conjecture, namely that the upper Minkowski dimension of a Besicovitch set in $\mathbb{R}^n$ is n. In this note we discuss this conjecture, giving a number of equivalent forms of it. We show that a natural finite field variant of it does hold. We also give some lower bounds.

(57) For a fixed $c > 0$ we construct an arbitrarily large set B of size n such that its sum set $B + B$ contains a convex sequence of size cn^2, answering a question of Hegarty.

(58) We investigate a discrete version of the Brunn-Minkowski inequality conjectured earlier by Matolcsi and Ruzsa, and we prove some special cases of it.

(59) We strengthen the volume inequalities for L_p zonoids of even isotropic measures and for their duals, which are due to Ball, Barthe and Lutwak, Yang, Zhang. Along the way, we prove a stronger version of the Brascamp-Lieb inequality for a family of functions that can approximate arbitrary well some Gaussians when equality holds. The special case $p = \infty$ yields a stability version of the reverse isoperimetric inequality for centrally symmetric bodies.

(60) A complete classification is established of Minkowski valuations on lattice polytopes that intertwine the special linear group over the integers and are translation invariant. In the contravariant case, the only such valuations are multiples of projection bodies. In the equivariant case, the only such valuations are generalized difference bodies combined with multiples of the newly defined discrete Steiner point.

(61) Let $F(x, y, z) = xy + z$. We consider some properties of expansion of the polynomial F in different settings, namely in the integers and in prime elds. The main results concern the question of covering $\{0, 1, \ldots, N\}$ (resp. F_p) by $A^2 + A$ with some thin sets A.

(62) It is established that there exists an absolute constant $c > 0$ such that for any finite set A of positive real numbers we have $|AA + A| >> |A|^{\frac{3}{2}+c}$.

(63) We prove two results about the cardinality (up to N) of multiplicative Sidon sets.

(64) We investigate lower bounds of the inverse Markov factor $M_{n,q}(K)$ for convex domains K in the complex plane. Finally, in this paper we prove a fully general result, valid for all convex domains K: with two positive constants $c(K)$ and $C(K)$ we always have $c(k)n/\log n < M_{n,q}(K) < C(K)n$.

(65) In this paper we give generalizations of earlier results of Erdos, Sarkozy and T. Sos about differences of values of a representation function of a set of nonnegative integers.

(66) In this paper, we study the existence of a minimal complement to a set W when W is eventually periodic or not. This partially answers a problem of Nathanson.