Global Dynamics of Differential Equations
Final report K 109782

The main aim of this project was the study of global type problems for dynamical systems generated by functional differential equations, ordinary differential equations, and partial differential equations. Most of the problems were motivated by the goal to understand the dynamics of real life phenomena.

During the supported period 2013-2018 (we asked one year extension) the research team published 70 research papers, most of them appeared in top mathematical journals. 60 papers acknowledged the financial support from NKFIH/OTKA. The research went according to the original research plan, only minor changes occurred.

It is impossible to include all results in this report. Therefore we highlight only a few of them representing the work of the team. However, it should be emphasized that many other interesting and important results are not mentioned below.

In the paper

B.Bánhelyi; T. Csendes; T. Krisztin; A. Neumaier: Global attractivity of the zero solution for Wright's equation.*SIAM J. Appl. Dyn. Syst.* 13 (2014), no. 1, 537-563

we studied the famous Wright's differential equation $x'(t) = -\alpha(e^{x(t-1)-1}$ with $\alpha > 0$. It was an open problem since 1955 that all solutions of the equation approach zero as $t \to \infty$ provided $\alpha \leq \pi/2$. We used deep analytical results to prove that it is sufficient to exclude the existence of slowly oscillating periodic solutions. Then we developed rigorous numerical tools to show nonexistence of slowly oscillating periodic solutions. The computer-assisted algorithms of the paper with a combination of analytical tools made it possible to handle successfully an infinite dimensional problem.

The authors of this paper received the prestigious *Moore prize in 2016* for a new approach in interval analysis to prove infinite dimensional attractivity results.

In fact we were able to show Wright's conjecture only for $\alpha < 1.5706$. We had ideas how to complete the proof. The missing part required an estimation close to the critical Hopf bifurcation point. Our idea was to construct a center manifold controlling the size of it. However, it turned out that this approach produced a neighborhood that was too small for rigorous computations. Nevertheless, the approach turned out to be useful in finite dimensions: In a paper (recently submitted) J. Dudás and T. Krisztin obtained the entire global attractivity region for a 3-dimensional delayed logistic difference equation. The computer-assisted algorithms developed in the above work by Bánhelyi, Csendes, Krisztin and Neumaier turned out to be an essential part of the recent complete proof for Wright's conjecture:

J.B. van den Berg; J. Jaquette: A proof of Wright's conjecture. to appear in *J. Differential Equations*, 2018.

Moreover, the same techniques were applied in the recent solution of another famous problem for the Wright's equation from 1962:

J. Jaquette, A proof of Jones' conjecture. Submitted. 2018.

In the work

T. Krisztin; G. Vas: The unstable set of a periodic orbit for delayed positive feedback. *J. Dynam. Differential Equations* 28 (2016), no. 3-4, 805-855.

we studied the differential delay equation $x'(t) = -\mu x(t) + f(x(t-1))$, where $\mu > 0$ and f is smooth and increasing, $f(\xi) - \mu\xi$ has exactly five zeros $\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$. So each ξ_k defines an equilibrium point of the semiflow generated by the delay differential equation. Earlier

[J. Dynam. Differential Equations 23 (2011), no. 4, 727-790] we showed that there are exactly two large amplitude periodic solutions, p and q, with the following properties: (i) p and q are hyperbolic and unstable, (ii) p has two simple real Floquet multipliers outside the unit circle, and q has one simple real Floquet multiplier outside the unit circle, and (iii) p and q have one or two zeros in every time interval of length 1. We described the global attractor and the role of the unstable manifolds of p and q within the global attractor. The paper shows that the unstable manifold of p is a 3-dimensional C^1-submanifold of the infinite dimensional phase space, and can be represented as a smooth graph. The same holds for connecting sets, i.e., those orbits with initial values in the unstable manifold which have the equilibrium ξ_{-2}, ξ_0 or ξ_2 as ω-limit sets. The connecting sets related analogously to ξ_{-1}, ξ_1 or q are shown to be two-dimensional submanifolds of the unstable manifold of p and homeomorphic to an open annulus in $\mathbb{R}^2$. Therefore the dynamics of the delay equation is almost completely described.

The paper

G. Vas: Configurations of periodic orbits for equations with delayed positive feedback. *J. Differential Equations* 262 (2017), no. 3, 1850-1896

discusses the number and the type of feasible configurations of the so-called large-amplitude periodic solutions for the scalar delay differential equation $x'(t) = -\mu x(t) + f(x(t-1))$, where $\mu > 0$ and f is smooth and increasing. A periodic solution is said to have large amplitude if it oscillates about at least two equilibrium points. The results impose strong restrictions on what type of large-amplitude periodic solutions for a given feedback function f. The paper gives not only a full characterization but also confirms the existence of all possible configurations of large-amplitude periodic solutions for any number $N \geq 2$ of unstable equilibria. In detail, it is shown that the number of different configurations coincides with the number of ways in which N symbols can be parenthesized, and that for any fixed parenthetical expression of N numbers there exists $\mu > 0$ and a C^1-smooth nondecreasing function f such that there are exactly N unstable equilibria and all the large-amplitude periodic solutions result from the parentheses in the given expression.

The work

T. Krisztin; A. Rezounenko: Parabolic partial differential equations with discrete state-dependent delay: classical solutions and solution manifold. *J. Differential Equations* 260 (2016), no. 5, 4454-4472

studies a class of parabolic partial differential equations with discrete state-dependent delays. A suitable framework is developed. A so-called solution manifold is defined which is a nonlinear analogue of the domain of the infinitesimal generator from the linear theory. For $\mathbb{R}^n$-valued differential equations with state-dependent delay H.-O. Walther showed that the corresponding solution manifold is a codimension n submanifold of $C^1([-h, 0], \mathbb{R}^n)$. Moreover, on the solution manifold the solutions define a semiflow with C^1-smooth solution operators. For the parabolic partial differential equation the solution manifold is not a finite codimensional manifold. Nevertheless, it turns out that the solution manifold is C^1-smooth, and this allows to prove that the evolution operators are C^1-smooth on the solution manifold. The result of the paper turned out to be applicable for PDEs with state dependent delay for linearized stability, for C^1-smooth local invariant manifolds.

In

T. Krisztin; H.-O. Walther, Smoothness issues in differential equations with state-dependent delay. *Rend. Istit. Mat. Univ. Trieste* 49 (2017), 95-112

we discuss the smoothness problem of the solution operator for differential equations with state-dependent delays. There is a so-called C^1-framework by Walther. This paper presents some examples to show that, in general, the smoothness cannot be expected more than C^1. In order to overcome the difficulties caused by the lack of smoothness, we introduce convolution and mollification techniques which allow to approximate the non-smooth evaluation map by means of smooth maps.

The paper

T. Faria; G. Röst: Persistence, permanence and global stability for an n-dimensional Nicholson system. *J. Dynam. Differential Equations* 26 (2014), no. 3, 723-744
studies the dynamical behaviors of some classes of Nicholson's blowflies systems with patch structure and multiple discrete delays. It is shown that if the spectral bound of a so called community matrix M is non-positive, then the population becomes extinct on each patch; however, the population uniformly persists if the spectral bound is positive. Moreover, if the population uniformly persists, then the model without delay has a unique positive equilibrium which is globally asymptotically stable under certain natural conditions.

L. Hatvani: On the global attractivity and asymptotic stability for autonomous systems of differential equations on the plane. *Proc. Amer. Math. Soc.* 145 (2017), no. 3, 1121-1129

proves some sufficient conditions for the global attractivity of the unique equilibrium point of a vector field in the plane. Earlier results by N. N. Krasovskii and C. Olech are improved for the case when the divergence of vector fields is everywhere less than zero. The case when the divergence is greater than zero is also studied.

L. Hatvani: On the damped harmonic oscillator with time dependent damping coefficient. *J. Dynam. Differential Equations* 30 (2018), no. 1, 25-37

studies a second-order linear differential equation with time-dependent damping coefficient, and proves asymptotic stability in the cases of small and large damping coefficient.

The paper

M. Polner; J.J.W. van der Vegt: A Hamiltonian vorticity-dilatation formulation of the compressible Euler equations. *Nonlinear Anal.* 109 (2014), 113-135

shows that using the Hodge decomposition on bounded domains the compressible Euler equations of gas dynamics can be reformulated using a density weighted vorticity and dilatation as primary variables, together with the entropy and density. This formulation is an extension to compressible flows of the well-known vorticity-stream function formulation of the incompressible Euler equations. The Hamiltonian and associated Poisson bracket for this new formulation of the compressible Euler equations are derived and extensive use is made of differential forms to highlight the mathematical structure of the equations. In order to deal with domains with boundaries also the Stokes-Dirac structure and the port-Hamiltonian formulation of the Euler equations in density weighted vorticity and dilatation variables are obtained.

Continuous finite element methods in space and discontinuous finite element methods in time for the discretization of neural field equations with space-dependent delays are constructed and analyzed in:

M. Polner; J.J.W. van der Vegt; S.A. van Gils: A space-time finite element method for neural field equations with transmission delays. *SIAM J. Sci. Comput.* 39 (2017), no. 5, B797-B818

A priori error estimates are established. The efficiency of the methods is demonstrated through a series of numerical results for neural field equations in one and two space dimensions.

In the paper

M.V. Barbarossa; M. Polner; G. Röst: Stability switches induced by immune system boosting in an SIRS model with discrete and distributed delays. *SIAM J. Appl. Math.* 77 (2017), no. 3, 905-923

the authors incorporate the host immune boosting into an epidemic model of the SIRS type where a time delay is used to represent the length of the immunity period. They establish a threshold condition for disease persistence in terms of the basic reproduction number. Interestingly, it is found that the endemic equilibrium exhibits stability switches as the delay increases. The local stability of the equilibrium is also analyzed as two important parameters vary.

Uniform disease persistence is investigated in

S.A. Gourley; G. Röst; H.R. Thieme: Uniform persistence in a model for bluetongue dynamics. *SIAM J. Math. Anal.* 46 (2014), no. 2, 1160-1184

for the time evolution of bluetongue, a viral disease in sheep and cattle that is spread by midges as vectors. The model is a system of several delay differential equations. As in many other infectious disease models, uniform disease persistence occurs if the basic disease reproduction number for the whole system, R_0, exceeds one. However, since bluetongue affects sheep much more severely than cattle, uniform disease persistence can occur in two different scenarios which are distinguished by the disease reproduction number for the cattle-midge-bluetongue system without sheep, $\tilde{R}_0$. If $R_0 > 1$ and $\tilde{R}_0 > 1$, bluetongue persists in cattle and midges even though it may eradicate the sheep, relying on cattle as a reservoir. If $R_0 > 1 > \tilde{R}_0$, bluetongue and all host and vector species coexist, and bluetongue does not eradicate the sheep because it cannot persist on midges and cattle alone. The two scenarios require different use of dynamical systems persistence theory.

The paper

D. Knipl; P. Pilarczyk; G. Röst: Rich bifurcation structure in a two-patch vaccination model. *SIAM J. Appl. Dyn. Syst.* 14 (2015), no. 2, 980-1017

considers he effect of vaccination by augmenting the susceptible and infectious classes in the usual SIS model of epidemiology by a class V of vaccinated individuals. The result may be called an SIVS model. The paper under review considers a model which couples two SIVS models to each other so as to describe two interacting populations. Under certain conditions, statements are proved concerning the number and stability of stationary solutions. Numerical simulations indicate that this model shows a bifurcation structure much more complicated than that of a single SIVS model. Rigorous results on the global dynamics complementing the simulations are obtained using techniques of interval arithmetic and the Conley index.

When the body gets infected by a pathogen the immune system develops pathogen-specific immunity. Induced immunity decays in time and years after recovery the host might become susceptible again. Exposure to the pathogen in the environment boosts the immune system thus prolonging the time in which a recovered individual is immune. Such an interplay of within host processes and population dynamics poses significant challenges in rigorous mathematical modeling of immuno-epidemiology. The paper

M.V. Barbarossa; G. Röst: Immuno-epidemiology of a population structured by immune status: a mathematical study of waning immunity and immune system boosting. *J. Math. Biol.* 71 (2015), no. 6-7, 1737-1770

proposes a framework to model SIRS dynamics, monitoring the immune status of individuals and including both waning immunity and immune system boosting. The model is formulated as a system of two ordinary differential equations (ODEs) coupled with a PDE. After showing existence and uniqueness of a classical solution, the authors investigate the local and the global asymptotic stability of the unique disease-free stationary solution. Under particular assumptions on the general model, known examples such as large systems of ODEs for SIRWS dynamics, as well as SIRS with constant delay are recovered.

Three members of the research team defended their PhD during the project: Diána Knipl (2014, supervisor Gergely Röst), Zsolt Vízi (2017, supervisor Gergely Röst), Ábel Garab (2014, supervisor Tibor Krisztin).