Final report of NKFIH project No. K 109240
Discrete optimization and its applications

The project proposal addressed major open problems in five specific areas of discrete optimization. We present the results of the project by grouping them into these five topics. Beside these results, five PhD theses were submitted and defended by the young researchers of the project (Bérczi-Kovács, Kaszanitzky, Cs. Király, Jankó and Tóthmérész).

During the funding period, 6 book chapters were published by the participants of this grant. A book chapter of the PI and Jordán on Graph connectivity augmentation appeared in the Handbook of Graph Theory. A book chapter of Z. Király in the topic of stable matchings was published in the Encyclopedia of Algorithms. A book chapter on the basics of combinatorial rigidity was published by Jordán in the MSJ Memoirs. Jordán (together with Whiteley) also wrote a new chapter on Global rigidity into the Handbook of Discrete and Computational Geometry. Another book chapter of Jordán on 2-dimensional global rigidity (with Jackson and Tanigawa) appeared in the Handbook of Geometric Constraint Systems Principles. Cs. Király (with Tanigawa) wrote a chapter in the latter Handbook on the rigidity of body-bar-hinge frameworks.

A trimester program on Combinatorial Optimization was organized at the Hausdorff Institute in Bonn (Germany) by the PI, Iwata, Könemann and Vygen. During the program four workshops (on connectivity, rigidity and submodularity, relaxations, and game theory) were organized where several members of the group were invited speakers or invited participants (Bérczi, Frank, Jankó, Jordán, Cs. Király, T. Király, Pap). There were several rigidity workshops in Lancaster, in Edinburgh organized by the ICMS, and in Banff organized by the BIRS where Jordán, Kaszanitzky and Cs. Király were invited speakers or invited participants. Jordán was also an invited speaker on the DIMACS Workshop on Distance Geometry at the Rutgers University (USA). Tóthmérész was an invited speaker on a BIRS-CMO Workshop on Sandpile Groups. Bérczi, T. Király and Pap were invied speakers of the Shonan Meeting on Combinatorial Optimization in Japan. The PI gave invited talks on the 4th ISCO in Vietri sul Mare, on the ECCO XXIX in Budapest, on the Tutte Centenary Conference held in Cambridge and on the Follow-up Workshop to Hausdorff Trimester Program on Combinatorial Optimization. T. Király gave a talk on SODA 2016 in Arlington (USA). Jordán gave invited talks on the LSE-QMUL Combinatorics Conference in London and on the Bond-node structures workshop in Lancaster. T. Király and Cs. Király gave talks at the European Conference on Combinatorics, Graph Theory and Applications. Most people from the group gave a talk on the 9th and 10th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications; furthermore, the latter one was co-organized by our group. Their results were also presented on the 20th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems and on the 25th Annual European Symposium on Algorithms. Bérczi, Kaszanitzky and Cs. Király were invited speakers at the conference Combinatorial Geometries: Matroids, Oriented Matroids and Applications in Marseille.

Between 2013-2018, the Egerváry Research Group organized an annual 5-day long workshop at the guesthouse of the Hungarian Academy of Sciences in Mátraháza. The aim of these meetings was to provide an inspiring environment for the participants to carry out particularly intensive joint research. The benefits of these meetings were not only that the participants could share and discuss their ideas on various problems, and they could even solve some of these problems on the spot, but these discussions often gave rise to exciting new questions initiating further successful researches. One concrete example is the 3-partite sequence of papers by Bérczi and Frank, appeared in MOR, which was initiated by a surprising observation made at such a workshop, but there are many other papers of the group which stemmed from initial researches at these meetings.

1 Network optimization

Baker and Norine introduced a graph-theoretic analogue of the algebraic geometric Riemann-Roch theory. A central notion in this theory is the rank of a divisor. Kiss and *Tóthmérész* proved that computing the rank of a divisor on a graph is NP-hard. Rotor-routing is a one-player game on a graph very similar to chip-firing. *Tóthmérész* showed that certain questions related to rotor-routing are decidable in polynomial time. The complexity of the reachability question in the chip-firing game is open in most cases. *Hujter*, Kiss and *Tóthmérész* showed that if the target distribution is recurrent, then the problem can be decided in polynomial time. They also showed that the reachability question of chip-firing games is in co-NP, moreover, for Eulerian digraphs, it is decidable in polynomial time. Kálmán defined the notion of hypertrees, that are generalizations of spanning trees of graphs to hypergraphs, and they also generalize the notion of break divisors. Using the notion of hypertrees, he defined the interior polynomial, a generalization of $T(x,1)$ to hypergraphs, where $T(x,y)$ is the Tutte polynomial. Kálmán and *Tóthmérész* gave an alternative definition to the interior polynomial by generalizing Bernardi's definition of the Tutte polynomial. As a biproduct, they obtain a method for dissecting the root polytope of a bipartite graph.

Bérczi-Kovács, Pedersen, Lucani and Fitzek proposed a design for wireless multi-layer multicast and analyzed its delay distribution and mean performance under various system conditions and presented a first implementation to verify their analysis and demonstrate the applicability of their approach. *Bérczi-Kovács* gave an improved, network size independent lower bound on the required field for failure protecting network code design, which also yields a faster algorithm for the problem. *Z. Király* and *Kovács* gave both randomized and deterministic algorithms for maximum throughput-achieving network code construction for the network code completion problem in the multicast case. They also introduced the problem of fixable pairs, and gave a sufficient condition for a set of coding coefficients to be fixable. Together with Pašić, Tapolcai, Babarczi, and Rónyai, they gave a polynomial algorithm for minimum cost diversity coding. *Bérczi-Kovács* and co-athors studied the problem of finding a pair of shortest disjoint paths that can be represented by only two forwarding table entries per destination. They showed that the underlying mathematical problem is NP-complete and presented heuristic algorithms and simulations that improve the known complexity bounds from cubic to the order of a single shortest path search. Feasibility of the proposed representations is confirmed by an extensive simulation study on real forwarding tables. *Bérczi-Kovács* and co-authors gave a linear time proactive recovery scheme against single edge failures for unicast connections in transport networks applying network coding at the source and the sink node only. *Bérczi-Kovács* and *Z. Király* give optimal and heuristic network coding algorithms and complexity result for multi-layer video streaming problems.

A path partition of a digraph D is called k-optimal if the sum of k-norms of its paths is minimal where the k-norm of a path P is $\min(|V(P)|, k)$. Berge's path partition conjecture claims that for every k-optimal path partition there are k disjoint stable sets orthogonal to the partition. For general digraphs the conjecture has been proven for $k = 1, 2, \lambda-1, \lambda$, where λ is the length of a longest path in the digraph. *Herskovics* proved the conjecture for $\lambda-2$ and $\lambda-3$. In a subsequent work, he combined his approach with that of an earlier paper to get an unified method which covers all known cases except one. *Z. Király* introduced the notion of nearly conservative digraphs and gave FPT algorithms for finding all-pairs shortest paths in them. *Bérczi* and Kobayashi gave the first polynomial time algorithm for the directed version of the Two Disjoint Shortest Paths Problem when the length of each edge is positive. They extended their result to the case when the instance has two terminal pairs and the number of paths is a fixed constant greater than two. They also showed that the undirected k disjoint shortest paths problem and the vertex-disjoint version of the directed k disjoint shortest paths problem can be solved in polynomial time if the input graph is planar and k is a fixed constant. *Bérczi* and Joó showed that for every (finite or infinite) cardinal $k > 0$ there is a cardinal $\ell(k)$ such that in every k-edge-colored tournament there exist disjoint vertex sets K, S with total size at most $\ell(k)$ so that every vertex v has a monochromatic path of length at most two from K to v or from v to S.

Erdős, *Frank* and Kun introduced the notion of sink-stable sets of a digraph and proved a min-max formula for the maximum cardinality of the union of k sink-stable sets. The results imply a recent min-max theorem of Abeledo and Atkinson on the Clar number of bipartite plane graphs and a sharpening of Minty's coloring theorem. They also exhibited a link to min-max results of Bessy and Thomassé and of Sebő on cyclic stable sets. *Bérczi-Kovács* and *Bernáth* proved that determining the Clar number of a 2-connected plane graph is NP-complete, and they gave an FPT algorithm for this problem. The parameter of the algorithm is the length of the shortest odd-join in the planar dual of the graph.

In an attempt to improve on Christofides' approximation guarantee for the Traveling Salesman Problem, Czeller and *Gy. Pap* considered a recent randomized approach by Gharan et al. that has a better-than-3/2 approximation ratio for graphic metric. Their contribution is that essentially the same algorithm also provides a better-than-3/2 ratio for a larger family of metric spaces, namely those that arise from a weighted undirected graph in which edge weights are taken from numbers between 1 and a fixed upper bound b. *Gy. Pap* also investigated the problem of finding better than 3/2-approximation for the TSP problem. He considered a dual counterpart of the problem, similarly to the Karr-Vempala approach. The equivalent question thus obtained is open; Gy. Pap considered special cases of this problem. *Gy. Pap* and Varnyú introduced a multi-agent variation of the traveling salesman problem, when the movement of the agents is required to satisfy node- and edge capacity constraints, and the goal is to find an optimal agency that has a as many agents as possible, and completes its set of tours in as small a time horizon as possible. They proved several lower and upper bounds in case of trees, and 3-regular 3-connected graphs.

Bernáth, Kobayashi, and Matsuoka solved the Generalized Terminal Backup Problem, which is the following. Given a set V and a subset T of V (T is called the terminal set), and a nonnegative requirement $r(t)$ for every terminal t, find a minimum cost graph such that the edge-connectivity between any terminal t and the rest of the terminal set $T - t$ is at least $r(t)$. The problem generalizes the weighted b-matching problem (where $T = V$), and the Terminal Backup Problem (where $r(t) = 1$ for every terminal t).The computational complexity of multicut-like problems may vary significantly depending on whether the terminals are fixed or not. *Bérczi*, *T. Király* and co-authors gave an algorithm with approximation factor strictly better than 2 for bicut, and showed that fixed-terminal double cut cannot be approximated to a factor better than 2 assuming the truth of the Unique Games Conjecture (UGC). They also showed an inapproximability factor of 3/2 for the global version. The approximability of the linear 3-cut problem in digraphs was wide open: the best known lower bound was 4/3 assuming the truth of the UGC, while the best known upper bound was 2 using a trivial algorithm. *Bérczi*, Chandrasekaran, *T. Király* and Madan completely closed this gap by presenting a $\sqrt{2}$-approximation algorithm and showing that this factor is tight assuming the truth of the UGC. They also considered the multiway cut problem, in which an undirected graph is given together with non-negative edge weights and a collection of terminal nodes, and the goal is to partition the node set of the graph into non-empty parts each containing exactly one terminal so that the total weight of the edges crossing the partition is minimized. For arbitrary k, the best-known approximation factor was 1.2965, while the best known inapproximability factor was 1.2. In their work they improved on the lower bound to 1.20016 by constructing an integrality gap instance for the CKR relaxation. *Bernáth*, Grappe and Szigeti proved a common generalization of the global edge-connectivity augmentation of a graph with partition constraints (solved by Bang-Jensen et al.) and the problem of covering a symmetric crossing supermodular set function (solved by Benczúr and the *PI*).

By extending an earlier result of Cranston, *Bérczi*, *Bernáth* and Vizer proved that every regular graph admits an antimagic labeling. A 2-matching in a bipartite graph $G = (S, T, E)$ is called V-free if it does not contain a path of length 2 with both ends in T. They proved a conjecture of Liang: if nodes in S have degree 4, and those in T have degree 3 then there exists a V-free 2-matching that covers the nodes in T. This conjecture is related to antimagic labelings. On the other hand, the authors showed that it is NP-complete to decide whether a bipartite graph $G = (S, T, E)$ admits a V-free 2-matching that covers the nodes in T.

Mészáros proved that if q is a prime power, D is a directed graph such that the underlying undirected graph is $(q-1, 1)$-partition-connected, then D contains q disjoint dijoins, thus settling a special case of the long-standing open conjecture of Woodall.

Motivated by a conjecture of Borradaile et al. on egalitarian strongly connected orientations, the *PI* and Murota developed a theory on the set of decreasingly minimal elements of an M-convex set. They not only proved the conjecture in an extended form but obtained large scale generalizations of several earlier, apparently unrelated results such as the ones of Megiddo on network flows, of Harvey-Ladner-Lovász-Tamir on semi-matchings, and of Levin and Onn on shifted matroid optimization. The two papers may be considered as a major step to develop new links between discrete convex analysis and combinatorial optimization.

T. Király and J. Pap introduced an LP-based model (a special case of the Generalized Nash Equilibrium Problem) where the interaction of players is limited to providing services to each other at fixed prices. This enabled the investigation of computational complexity as a function of the structure of provider-customer relationships. The model guarantees the existence of equilibria with polynomial bit-size. In a joint paper with Japanese co-authors, *T. Király* proved that diamond-submodular functions (submodular functions given on the direct product of diamond lattices) can be minimized in polynomial time. *Z. Király* and *Kisfaludi-Bak* gave and algorithm that calculates $3 \log n$ bit addresses for the nodes appropriately for greedy navigation. *Györgyi*, *Hujter* and *Kisfaludi-Bak* established an upper bound on the number of touching pairs in a set of planar curves where each pair meets (crosses or touches) exactly once. *Bernáth* gave a short proof of a theorem of Benczúr and Frank on covering a symmetric crossing-supermodular function with graph edges. Gross, Shokrieh and *Tóthmérész* introduce the notion of semibreak divisors on metric graphs and prove that every effective divisor class of degree at most the genus has a semibreak divisor representative. This generalizes the notion of break divisors which are known to be a highly useful concept in tropical geometry, and it also has connections to other parts of mathematics. Their proof uses submodular minimization. Then semibreak divisors are used to establish some properties of effective loci inside Picard groups of metric graphs. Suppose that, given some functions f and g, we want to order the vertices of a graph such that every vertex v is preceded by at least $f(v)$ of its neighbors and succeeded by at least $g(v)$ of its neighbors. *Bérczi*, Iwata, Kato and Yamaguchi gave an algorithm for making a bipartite graph DM-irreducible by adding a minimum number of new edges. *Z. Király* and Pálvölgyi showed that this problem is solvable when $fg \equiv 0$ and is NP-hard even in the simple case where $f \equiv g \equiv 2$. Consider all weightings of the vertices of a simple graph G with real numbers whose total sum is non-negative. How many edges of G have endpoints with a non-negative sum? *Z. Király* and his students considered the minimum number of such edges over all such weightings as a graph parameter. Computing this parameter had been shown to be NP-hard but they gave a polynomial algorithm to compute the minimum of this parameter over realizations of a given degree sequence. They also completely determined the minimum and maximum value of this parameter for regular graphs.

Motivated by a real life application, *Bérczi* and *Jüttner* investigated route-planning problems for public road surveillance vehicles. They considered several variants depending on the allowed visiting patterns of each road segment and on whether or not the roads are preassigned to the surveillance vehicles. Within the framework of a project initiated by the Hungarian Public Road Nonprofit Private Limited Company, *Bérczi* and *Jüttner* gave a method for efficiently solve the problem of routing of vehicles surveying large public road networks. The capability of the solution is demonstrated on the Hungarian non-urban public road network. They also examined a stochastic public transport routing problem, when the arrival and travel time of the vehicles are not deterministic and the goal is to maximize the probability of arriving in time. Instead of a simple route, the solution is capable of providing a routing policy, which is a multiple choice and time dependent itinerary.

2 Trees and arborescences

Cs. Király gave an arborescence-packing theorem on matroid-rooted arborescences genaralizing the results of Kamiyama, Katoh and Takizawa and of D. de Gevigney, Nguyen and Szigeti. *Bérczi*, *T. Király* and Kobayashi proved a theorem on covering intersecting bi-set families under matroid constraints. Their result can be considered as a common generalization of *Cs. Király*'s result and of a result of *Bérczi* and the *PI*. In a subsequent work, they presented a polynomial-time algorithm for solving the weighted version of the problem. *Cs. Király*, with his co-authors, extended the existing arborescence-packing results to dypergraphs with a simple technique that also implies an algorithm to the weighted case of these problems. They also verified a conjecture of *Bérczi* and *Frank* on packing spanning arborescences with matroid constraints for several important matroid classes such as transversal matroids and graphic matroids. On the other hand, they showed that the conjecture is false in general, furthermore, the corresponding decision problem is NP-complete. *Cs. Király* and Szigeti further extended the list of previous results on packing arborescences under matroid constraints. In a subsequent work, they showed with Tanigawa that the arc sets of such packings can be considered as a matroid intersection which gave a simpler proof for the above result and an algorithm for its weighted case. This result used some ideas from the above work of *Bérczi*, *T. Király* and Kobayashi and defined a matroid by using a bi-set function.

Bernáth and *Gy. Pap* constructed a polynomial time algorithm for an input digraph D and a positive integer k to find a minimum cardinality set of arcs in a digraph the deletion of which results in a digraph that has no k disjoint arborescences where the root node of the arborescences is not fixed. They also gave a polynomial algorithm for the problem of finding a smallest possible subset F of arcs in an edge-weighted digraph such that F intersects every minimum weight arborescence. *Bernáth* and *T. Király* studied the problem of finding a minimum cardinality subset of arcs that contains at least one arc from every minimum cost k-arborescence and gave an algorithm for general k that has polynomial running time if k is fixed. *Bérczi*, *Bernáth*, *T. Király* and *Gy. Pap* considered the problem of finding a minimum weight transversal of a family F, where F is the set of optimal solutions of some combinatorial problem, for example, optimal k-spanning trees, optimal k-arborescences, or optimal k-braids.

Bernáth and *Z. Király* considered 44 types of path/circuit/forest/spanning tree packing/partitioning/covering problems and examined their complexity. For some problems that were found to be NP-hard, e.g. for the problem of partitioning the edge set of a graph to 2 paths, they also proved the NP-hardness for planar graphs.

Given a graph and numbers on the vertices, a natural (but NP-hard) problem is to decide whether a spanning tree exists with degrees at least the prescribed ones. A sufficient condition for the general case and a good characterization for a special case were known with lack of any algorithm. *Z. Király* gave polynomial algorithms for checking these conditions and – in case these are satisfied – for constructing the tree. *Bérczi* and Kobayashi showed that it can be decided in polynomial time whether a graph is a cycle-plus-triangles graph. *Bérczi*, *Z. Király*, Liu and Miklós showed that if two tree degree sequences do not have common leaves then they always have edge-disjoint caterpillar realizations. By using a probabilistic method, they also verified that two tree degree sequences always have edge-disjoint realizations if each vertex is a leaf in at least one of the trees.

In an earlier paper by the *PI* and *Jordán*, a min-max theorem was developed on optimally covering a so-called supermodular bi-set function by digraphs. This framework characteristically differs from previous models since it solves such cardinality optimization problems for which the corresponding weighted versions are NP-hard. It turned out that even the cardinality optimization includes NP-hard special cases when the digraph is required to be simple. Still, *Bérczi* and *Frank*, in a series of three papers, described special cases where simplicity can successfully be treated. In the first part, they developed a min-max theorem on covering an intersecting supermodular function with a simple degree-constrained bipartite graph and showed that several, apparently independent problems can be answered by using this general framework. One application is a new theorem on disjoint branchings, which provides necessary and sufficient condition for the existence of k disjoint branchings with prescribed sizes. As another consequence, they showed that even an extension of Ryser's maximum term rank problem nicely fits in the new framework. In the second paper of the series, they gave a matroidal extension of the framework and worked out the augmentation version of the problem. The third paper presents a generalization of a recent result of Hong, Liu, and Lai on characterizing the degree-sequences of simple strongly connected directed graphs, and provided a characterization for degree-sequences of simple k-node-connected digraphs.

3 Rigid structures

Fekete, *Jordán* and *Kaszanitzky* gave a necessary and sufficient condition for the existence of an infinitesimally rigid 2-dimensional bar-and-joint framework (G, p), in which the positions of two given nodes coincide. They also determined the rank function of the corresponding modified generic rigidity matroid on ground-set E. The results lead to efficient algorithms for computing the rank of G in the matroid.

Jordán and *Kaszanitzky* developed a new inductive construction of 4-regular $(1, 3)$-tight hypergraphs and used it to solve problems in combinatorial rigidity. They gave a combinatorial characterization of generically projectively rigid hypergraphs on the projective line. This result also implies an inductive construction of generically minimally affinely rigid hypergraphs in the plane. Based on the rank function of the corresponding count matroid on the edge set of $\mathcal{H}$, they obtained combinatorial proofs for some sufficient conditions for the generic affine rigidity of hypergraphs.

The combinatorial structure of a body-hinge framework can be encoded by a multigraph H, in which the vertices correspond to the bodies and the edges correspond to the hinges. *Jordán*, *Cs. Király* and Tanigawa proved that a generic body-hinge realization of a multigraph H is globally rigid in $\mathbb{R}^d$, for all $d \geq 3$, if and only if $(D-1)H - e$ contains D edge-disjoint spanning trees for all edges e of $(D-1)H$ where $D = \binom{d+1}{2}$. This implies an affirmative answer to a conjecture of Connelly, Whiteley, and *Jordán*. They also considered bar-joint frameworks and showed, for each $d \geq 3$, an infinite family of graphs satisfying Hendrickson's well-known necessary conditions for generic global rigidity in $\mathbb{R}^d$ (that is, $(d+1)$-connectivity and redundant rigidity) which are not generically globally rigid in $\mathbb{R}^d$. The existence of these families disproves a number of conjectures, due to Connelly, Connelly and Whiteley and Tanigawa, respectively. *Jordán* and Tanigawa characterized when the edges of a triangulated polyhedron with at least one bracing edge form a globally rigid graph. They introduced the notion of non-degenerate stresses and proved that non-trivial vertex splitting maintains global rigidity when the framework admits a non-degenerate stress.

A graph $G = (V, E)$ is called k-rigid in $\mathbb{R}^d$ if $|V| > k$ and after deleting any set of at most $k-1$ vertices the resulting graph is rigid in $\mathbb{R}^d$. A k-rigid graph G is called minimally k-rigid if the omission of an arbitrary edge results in a graph that is not k-rigid. *Kaszanitzky* and *Cs. Király* gave a lower bound for the number of edges in a k-rigid graph for arbitrary values of k and d and showed its sharpness for the cases where $k = 2$ and d is arbitrary and where $k = d = 3$. They also provided a sharp upper bound for the number of edges of minimally k-rigid graphs in $\mathbb{R}^d$ for all k.

Kaszanitzky and Schulze derived new symmetry-extended counting conditions for a picture with a nontrivial symmetry group in an arbitrary dimension to be minimally flat. They also gave sufficient connectivity conditions for the rigidity of symmetric frameworks in the plane. Together with Tanigawa, they gave a combinatorial characterization of infinite periodic graphs that are globally rigid in the plane. *Kaszanitzky*, *Cs. Király* and Schulze gave a sufficient condition for the fixed-lattice periodic global rigidity and characterized the fixed-lattice periodic global rigidity of periodic body-bar frameworks.

A graph G is called (k, ℓ)-rigid if it has a spanning (k, ℓ)-tight subgraph, and G is (k, ℓ)-redundant if $G - e$ is (k, ℓ)-rigid for any single edge e. *Cs. Király* gave a polynomial algorithm for the following augmentation problem for $\ell \leq 3/2k$. Given a (k, ℓ)-tight graph G, find a graph H with minimum number of edges, such that $G + H$ is (k, ℓ)-redundant. His algorithm can be extended for (k, ℓ)-rigid inputs when $\ell \leq k$.

4 Application of polynomial matrices

Jackson, *Jordán* and Tanigawa considered the problems of completing a low-rank positive semidefinite square matrix M or a low-rank rectangular matrix N from a given subset of their entries. They presented combinatorial characterizations of local and global (unique) completability for special families of graphs. They characterized local and global completability in all dimensions for cluster graphs, i.e. graphs which can be obtained from disjoint complete graphs by adding a set of independent edges. These results correspond to theorems for body-bar frameworks in rigidity theory. They also provided a characterization of two-dimensional local completability of planar bipartite graphs, which leads to a characterization of two-dimensional local completability in the rectangular matrix model when the underlying bipartite graph is planar. These results are based on new observations that certain graph operations preserve local or global completability, as well as on a further connection between rigidity and completability. They also proved that a rank condition on the completability stress matrix of a graph is a sufficient condition for global completability. This verifies a conjecture of Singer and Cucuringu. In a subsequent work, they obtained further results on the upper problems. They showed that the unique completability testing of rectangular matrices is a special case of the unique completability testing of positive semidefinite matrices. They proved that a generic partially filled matrix is globally uniquely completable if any principal minor of size $n - 1$ is locally uniquely completable. These results are based on new geometric observations that extend similar results of the theory of rigid frameworks. They also gave an example showing that global completability is not a generic property in two dimensions. They provided sufficient conditions for local and global unique completability of a square matrix of size n by proving tight lower (resp. upper) bounds on the minimum number of known entries per row (on the total number of unknown entries, resp.) as a function of n and the rank.

T. Király and J. Pap proved an extension of Lehman's theorem on minimally nonideal clutters that is a common generalization with the characterization of minimally imperfect clutters. They also showed how to extend the notion of idealness to unit-increasing set functions, in a way that is compatible with minors and blocking operations. *T. Király* proved that an integer polyhedron in the hyperplane of 0-sum vectors is a base polyhedron if and only if it has the linking property. The result implies that an integer polyhedron has the strong linking property, as defined in [*Frank*, *T. Király*, A survey on covering supermodular functions, 2009], if and only if it is a generalized polymatroid.

Hujter and *Tóthmérész* gave a new proof for the Riemann-Roch theorem of graphs that can be naturally extended to give a Riemann-Roch inequality for Eulerian digraphs.

5 Stable matchings

Fleiner described a network flow model that is related to ordinary network flows the same way as stable matchings are related to maximum matchings in bipartite graphs. He proved that there always exists a stable flow and generalized the lattice structure of stable marriages to stable flows.

Fleiner and *Jankó* built an abstract model, closely related to the stable marriage problem and motivated by Hungarian college admissions. They studied different stability notions and showed that an extension of the lattice property of stable marriages holds in these more general settings, even if the choice function on one side is not path independent. The main virtue of their work is that it exhibits practical, interesting examples, where non-path independent choice functions play a role, and proves various stability-related results.

Peer evaluation of research grant applications is a crucial step in the funding decisions of many science funding agencies. Funding bodies take various measures to increase the independence and quality of his process, sometimes leading to difficult combinatorial problems. *Fleiner* and his coauthors proposed a novel method based on network flow theory to find assignments of evaluators to grant applications that obey the rules formulated by the Slovak Research and Development Agency.

Fleiner and *Jankó* generalized a recent result of Aharoni, Berger and Gorelik. They showed that on two posets there always exists a weighted kernel, and the kernels form a lattice under a certain natural order.

Cechlárová, *Fleiner* and *Jankó* studied a real-life modification of the classical house-swapping problem that is motivated by the existence of engaged pairs and divorcing couples. They showed that the problem of finding a new house for the maximum number of agents is inapproximable, but fixed-parameter tractable.

Fleiner, *Jankó*, Tamura and Teytelboym consider a model over trading networks, define two new stability concepts, called trail stability and full trail stability. They show existence and lattice structure, and describe the relationships between various other concepts. They generalize the rural hospitals theorem, show strategy-proofness and comparative statics of firm entry and exit.

T. Király and *Mészáros-Karkus* gave a polynomial-time algorithm for finding a strongly popular b-matching in a variant of the student-project allocation problem. Mészáros-Karkus also proved that the b-way stable l-way exchange problem is NP-complete for any $b \geq 2$, $l \geq 3$.

6 Other results

Based on a previous result of Fleiner, *Fleiner* and Wiener gave a new proof for a signed graph generalization of Brooks theorem. Motivated by Ryser's conjecture on r-partite intersecting hypergraphs, *Z. Király* and *Tóthmérész* proved partial results on the coverability of edge-colored complete graphs by monochromatic components. Gyárfás and *Z. Király* proved characterization on the minimum number of monochromatic components covering of a k-colored complete r-uniform (r, ℓ)-partite hypergraph for many cases of the triple (k, r, ℓ). *Jüttner* and *Madarasi* gave efficient methods to various versions of the subgraph isomoprhism problem. The algorithms significantly outperforms the existing solutions, especially on instances coming from real life biological problems.

Z. Király with Füredi and Gyárfás studied intersecting set pair systems $\{A_i, B_i\}$ (where $|A_i| = |B_i| = n$, $|A_i \cap B_i| = 0$ and $|A_i \cap B_j| > 0$ for all $i \neq j$) with restrictions on the sizes of possible intersections. They showed that if we require $|A_i \cap B_j| = 1$ for all $i \neq j$, then the size of the system can be exponential, however if we require $|A_i \cap A_j| = 1$ for all $i \neq j$, then size is at most $n^2 + n + 1$ and this is essentially sharp.

Z. Király with his students studied the behavior of a new graph parameter $\mu(G)$ which is the minimum cardinality of $F \subseteq E(G)$, so that $G - F$ does not have a spanning subgraph with K_2 and cycle components. Given a graphical degree sequence, they focused the minimum/maximum value of $\mu(G)$ over the realizations of the sequence. The main results are a polynomial algorithm for computing the minimum, and a complete determination of the minimum and of the maximum for all regular degree sequences.

Bérczi and *Bérczi-Kovács* gave a polynomial time algorithm which constructs a body-minimal representation of pure Horn functions. The algorithm in fact finds the so-called GD basis of the function in question. *Jüttner* and *Madarasi* presented a refined approach to using column generation to solve specific type of large integer problems. The technique is demonstrated on the Generalized Assignment problem and Parallel Machine Scheduling problem as two reference applications.