Rigorous theory of hyperbolic systems with singularities (with an emphasis of questions of physics)

Closing report for the NKFI/OTKA K-104745 proposal

Domokos Szász

1 Introduction

The key goal of our research plan was – also reflected in the title of the project – the study of hyperbolic systems with singularities. In parts 2.1, 3 and 4 the fundamental objects of our research were hyperbolic billiards whereas in part 2.2 iterated function systems (IFS). In billiard theory the main circles of questions were around multi-dimensional billiards (2.1.1), intermittent behavior in planar billiards (2.1.2), non-homogeneous models (3) and energy transfer (4) while in the theory of ITF (2.2) the exact dimensionality of the interesting measures and, moreover, the validity of an analogue of Ledrappier-Young formula for them. Beyond the above topics we also have important results in the basic theory of planar hyperbolic billiards collected in the separate subsection 2.1.3; there it is also mentioned how they are related to other topics of our project. In all topics we could reach essential new results and our group belongs to the top of international research. In particular, our publications have appeared or accepted in much prestigious international periodicals.

2 Hyperbolic systems with singularities

2.1 Hyperbolic billiards

By the time our research project started, the theory of hyperbolic billiards had been developed significantly. Of crucial importance is the monograph [CM06] which provides a framework for studying *two dimensional dispersing billiards* using, in particular, the technique of *standard pairs*. Our previous work had made essential contribution to certain fields, specifically to the geometry and the dynamics of *multi-dimensional dispersing billiards* ([BCSzT03], [BT08], [BT12]) and to the emergence of *nonstandard probabilistic behavior* (corresponding to *superdiffusion*) in two dimensional billiard models with intermittency ([BG06], [SzV07], [BCD11]).

In course of the research performed within the framework of this project, our goal was to obtain an even finer understanding of hyperbolic billiards, strongly motivated by questions that arise in *non-homogeneous models* and those with a *separation of timescales*. Accordingly, the results that we are mentioning here are connected to the other sections of the present report. In particular, several results are also of preparatory nature to our ongoing big project on executing the rare interaction limit in a Hamiltonian system of particles in connection with the Gaspard-Gilbert strategy.

2.1.1 Statistical properties of multi-dimensional billiards

The lack of multi-dimensional model where the complexity hypothesis of [BT08] could be verified, it is a widely accepted approach in some branches of mathematics, like e. g. number theory, to go to the direction of conditional results. In that line the British physicist Dettmann formulated three hypotheses that would govern the asymptotic behavior of multi-dimensional Lorentz precesses with infinite horizon, cf. [D12]. In [NSzV14] two geometric conjectures of Dettmann are proved which concern the decay of the free flight time in high dimensional periodic Lorentz processes. This constitutes a major step in understanding the statistical properties of high dimensional billiards with infinite horizon. In particular, superdiffusion is expected if and only if the infinite horizon is of maximal dimension.

As mentioned above, further *development of the technique of standard pairs* has been one of our principal goals. Although we have made good progress about the multi-dimensional generalization ([BN16a]), this project is still to be completed.

Finally we mention that, with the theory of two-dimensional dispersing billiards having been more or less well understood, the goal of [Sz17] is to survey the key results of the theory of multi-dimensional dispersing billiards and highlight some central problems which deserve particular attention and efforts in this case.

2.1.2 Intermittent planar hyperbolic billiards

We have contributed to the clarification of *mixing rates of billiard type flows with intermittent behavior*. In [BN16b] it is proved that the continuous time dynamics of the system of two falling balls is rapid mixing for almost every value of the mass parameter in a certain interval. In addition to showing that the flow mixes faster than the map (cf. [BBN12]), this work is also of methodological interest given the lack of contact structure for the system of falling balls. The recent manuscript [BBM17] proves sharp results on polynomial decay of correlations for nonuniformly hyperbolic flows. Applications include intermittent solenoidal flows and various Lorentz gas models, in particular the infinite horizon Lorentz gas, settling this way a long standing conjecture.

As for the *finer analysis of superdiffusive phenomena*, in [BCD17] the emergence of an anomalous behavior is proved concerning the second moments in dispersing billiards with cusps: the second moments of the appropriately scaled Birkhoff sums converge to a value which is twice the second moment of the limit distribution. This result, in addition to its dynamics interest, is relevant from a purely probabilistic perspective, and answers one of the questions explicitly formulated in our research plan.

2.1.3 Basic theory of hyperbolic billiards

We generalized the technique of standard pairs to billiard flows in [BNSzT18], extending this way the breakthrough result of [BDL15] to a setting which seems appropriately suited for our long term project initiated in [BGNSzT17]. The related manuscript [T17] studies a notion of generalized Hölder continuity for functions on the Euclidean space, and show that a wide class of functions has this property automatically. This is a general result in geometric measure theory, with direct application in the theory of dynamical systems, including our work [BNSzT18] on dispersing billiard flows. Motivated by another problem exhibiting a separation of time scales, [DN16b] obtains further important estimates on the continuous time evolution of standard pairs. We note that this results is methodologically related to [BNSzT18]. For further details on these projects, see Section 4

As said, the problem of *complexity* is a central issue in the studies of hyperbolic systems with singularities. In [ST14] we analyze this question for planar dispersing billiards with corner points, and provide a satisfactory solution by proving the growth lemma. This result is is most relevant for applications, especially as the fast dynamics of the model introduced in [BGNSzT17] is a dispersing billiard with corner points.

Some further important results concern the *local central limit theorem for flows* and its applications. In [DN16a] – which proves that the particle density profile in a long Lorentz tube is governed by the heat equation, possibly with non-equilibrium boundary conditions – a major intermediate result is the local central limit theorem for the Sinai billiard flow. [DN17b] formulates some abstract conditions under which a suspension flow satisfies the local central limit theorem, and checks the validity of these conditions for several systems, including Sinai billiards with finite horizon. [DN17a] investigates connections of the local central limit theorem to problems in infinite ergodic theory. See Section 4 for further details.

Markov partitions, a notion whose systematic foundations were laid down by Sinai, have been a fundamental tool applied to a variety of problems in the theory of hyperbolic dynamical systems. For singular systems, however, this method turned out to be much cumbersome. Therefore there arose several modifications of the method: Markov sieves, Markov towers, standard pairs among them. The goal of [Sz18], based on the author's invited Science Lecture at Yakov Sinai's 2014 Abel Prize presentation, is to survey the main steps of the aforementioned developments and the main achievements based upon them. The work also summarizes the impact of Sinai's works on deriving laws of statistical physics (ergodic hypothesis, diffusion) from microscopic assumptions. A further goal is to formulate some central open questions of the theory.

2.2 Iterated function systems

The most important aim of our research covered in this point, was to get a far better understanding of the dimension theory of self-affine Iterated Function Systems.

Fractal sets, appearing all around in science and nature, are sets with unusual geometric structure which are obtained in many cases as the attractor of some hyperbolic dynamical systems. The most regular fractals are the self-affine (in particular self-similar) sets. An affine Iterated Function system (IFS) is a list $\mathcal{F} := \{f_i : x \mapsto A_i x + t_i\}_{i=1}^{m}$ of contracting affine maps on $\mathbb{R}^d$. We consider the case when $d \geq 2$. In the dimension theory of self-affine sets and self-affine measures (the unique non-empty compact set Λ such that

$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda)$ and for probabilities $(p_i)_{i=1}^{m}$ the unique measure μ such that $\mu = \sum_{i=1}^{m} p_i \mu \circ f_i^{-1}$), our main goals were

(a) to show that the self-affine measures are exact dimensional and

(b) to find the analogue of the Ledrappier-Young formula [LY85, Theorem C'].

We have reached all of our goals set in the corresponding part of the Research Plan and additionally we obtained related results for random iterated function systems. Our research related to the self-affine fractals contributed significantly to the very rapid development of this research field.

Related to research goal (a) above: In the paper [B15], the exact dimensionality was proven for a special family of affine IFSs on the plane and we gave an adapted version of the Ledrappier-Young formula. Later, in [BK17], the exact dimensionality was shown in full generality for planar affine IFSs. Moreover, under some conditions (simple Lyapunov spectrum) we gave the Ledrappier-Young formula and we showed exact dimensionality of self-affine measures in higher dimensions. These results can be considered as the generalisations of the results of Feng and Hu [FH09]. In both cases it turned out that stationary measures of group actions on the Grassmannian manifolds (i.e. Furstenberg measures) play extraordinary important roles.

Related to research goal (b) above: We gave several applications of the Ledrappier-Young formula. In [BR18] and [BKK17], we showed that there exist open sets of matrices such that for almost every matrices in the set, if the images of the set are disjoint (i.e. ($f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$ for $i \neq j$) the dimension of the self-affine sets is equal to the affinity dimension (which was introduced by Falconer [F88]). Moreover, in [BR18], for the family of IFSs, we showed that there exists an invariant measure with maximal Hausdorff dimension, which is a Gibbs measure. On the other hand, we also studied the case, when there are overlaps between the images. In [BRS16], by combining the results of Hochman [H14] and Feng and Hu [FH09], we studied the dimension sets and measures with diagonal matrices on the plane, allowing overlaps. As a continuation of this work, in [BRS17], we studied the dimension sets and measures with lower triangular matrices. The methods were based on the Ledrappier-Young formula and the transversality between the images of the self-affine set. In [S14], a review was given on the dimension theory of self-affine sets and measures and the recent progress on almost self-affine sets.

Furthermore, we studied three more problems in the topic of the deterministic IFSs.

(1) In [BKK18], we studied the multifractal spectrum of the pointwise Hölder exponents of curves, generated by affine IFS. (2) In [BK15], we studied the absolute continuity of a stationary measure with respect to IFSs with similarities on the real line, where the probabilities are place-dependent (i.e. Blackwell measure). This measure plays important role in the theory of hidden Markov-chains. (3) Finally, in [B17], we studied the non-linear images of self-similar sets in $\mathbb{R}^3$ to $\mathbb{R}$. We showed that if the dimension of such set is large enough, the dimension of the image is 1. We applied the result to study the distance set and the algebraic product of self-similar sets.

Additionally we also studied two related problems for random fractals. The first one is related to the Fractal percolation sets and the second one was an application of fractals theory for better understanding of Internet traffic.

(1) One of the most famous random fractals is the so called fractal percolation Cantor set. In [RS14a] we considered the two dimensional fractal percolations and we prove that the dimension of their orthogonal projections is the same in each direction. On the plane, M. Rams and K. Simon proved that for almost all realization of a fractal percolation set, if the dimension is greater than one then the interior f all orthogonal projections are non-empty. We extended this result here to higher dimension in [SV14]. In [RS14b] we presented a survey article about the recent results related to the projections of fractal percolation sets.

(2) M. Rams and J.L. Vehel described the multifractal structure of the RENO Internet traffic. This is an old version of the Internet. In our paper [SMKM17] we extended their results to a modern version of part of the Internet traffic generated by the so called cubic TCP.

3 Beyond homogeneity

We solved all problems raised in Section 3.1 of the research plan earlier than we expected in [NSzV12] (since this paper appeared in 2012, strictly speaking is not part of the present report).

Section 3.2 of the research plan is about systems with spatial inhomogeneity. One such model is an alternating chain of localized billiard balls and pistons, the study of which we initiated in [BGNSzT17].

Specifically, we studied the geometry of the model and made predictions about the behavior in the rare interaction limit (cf. Section 4). In [N16], we proved local thermodynamic equilibrium for systems of interacting particles (a related stochastic model) with spatial inhomogeneity. Namely, 1) the hydrodynamic limit is taken in models with some special inhomogeneity in high dimensions and 2) the non-equilibrium steady state is considered in one dimension with general inhomogeneity. Here, the inhomogeneity means either space dependent interaction rate, or varying degrees of freedom of the subsystems (e.g. as in [BGNSzT17]).

We have proved the local limit theorem and discussed its relation to mixing for many hyperbolic flows in [DN17b] (finite measure case) and [DN17a] (infinite measure case). This can be a starting point of future research on the mixing properties of infinite inhomogeneous systems which can be well approximated by homogeneous ones (e.g. billiards with local perturbations, ping-pong models).

Related to this topic is another set of results on globally coupled identical circle maps. One line of our research considers the case of finitely many coupled maps. In [SB16] and [S17], we consider the special case when the circle map is the doubling map. In this setting, two distinct bifurcation values of the coupling strength have been identified in the literature, corresponding to the emergence of contracting directions and, specifically for $N = 3$ units, to the loss of ergodicity. In [SB16], we improve these results and provide an interpretation of the observed dynamical phenomena in terms of the synchronization of the sites. For $N = 4$, we prove the absence of ergodicity for sufficiently strong coupling in [S17].

Our second line of research deals with the continuum limit of our coupled map system. In [SB16], we consider the special case when the circle map is the doubling map. We show that the unique invariant density exponentially attracts all initial distributions considered, provided that the coupling is sufficiently weak. We generalize this result in [BKST17] for a wider class of individual dynamics, after having shown that for sufficiently small coupling strength the system admits a unique absolutely continuous invariant distribution, which depends (Lipschitz continuously) on the coupling strength ε. For sufficiently strong coupling we prove that a wide class of initial measures approach a point mass with support moving chaotically on the circle (we show this for the case of the doubling map in [SB16] and more generally in [BKST17]).

4 Energy transfer and separation of time scales

This part of our research, in general, aims at understanding phenomena of heat conduction. The main long term perspective is to prove the validity of Fourier's law in extended systems with deterministic dynamics being as physically realistic as possible. As outlined in the research plan, we have two main points of attack: 1.) convergence of deterministic models to stochastic processes in a time scaling limit, while the system size is fixed, 2.) thermodynamic behaviour of the appearing stochastic interacting particle systems as the system size goes to infinity. We reached several results in both of these.

In the acclaimed program suggested by the physicists Gaspard and Gilbert [GG08] part 1.) of the above formulated strategy belongs to the theory of dynamical systems while its part 2.) to stochastics. We have two important new results related to part 1.). In both of them we consider the rare interaction limit of [GG08].

- In [DN16b] we prove that the time of the first collision between two particles in a Sinai billiard table converges weakly to an exponential distribution when time is rescaled by the inverse of the radius of the particles. This result can be interpreted as a first step in studying the energy evolution of hard ball systems in the rare interaction limit.

- In [BGNSzT17] we introduced a Hamiltonian, quasi one-dimensional model of interacting particles without mass transport where the rigorous study of energy transport is a realistic task. We also calculated, in a heuristic way, the transition kernel of the Markov jump process of energy exchanges arising in the rare interaction limit and, moreover, executed a positive statistical test. The manuscript which proves the rare interaction limit rigorously, is under preparation. A key auxiliary step is done in the work [BNSzT18] mentioned in Section 2.1.

In [DN16a] we prove that the particle density profile in a long Lorentz tube is governed by the heat equation, possibly with non-equilibrium boundary conditions. A major intermediate result is the local central limit theorem for the Sinai billiard flow.

In [LNY16] local thermodynamic equilibrium in the non-equilibrium steady state is proved for a stochastic system of interacting particles. The particular model is a stochastic version of the mechanical systems studied by Eckmann-Young [EY05] and Lin-Young [LY10]. The work [N16] also belongs here, see Section 3.

Closely related to the problem of heat conduction is the work [DN17b], in which we formulate some abstract conditions under which a suspension flow satisfies the local central limit theorem. We check the validity of these conditions for several systems, such as reward renewal processes, Axiom A flows, and systems admitting Young towers (Sinai billiard with finite horizon, suspensions over Pomeau-Manneville maps, geometric Lorenz attractor).

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