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PROBLEMS IN DISCRETE AND CONVEX GEOMETRY — FINAL REPORT —

FOR THE
NATIONAL RESEARCH, DEVELOPMENT AND INNOVATION OFFICE

The project involved the study of various topics in discrete and convex geometry. We considered covering problems (see Section 1), a quantitative Helly type problem (see Section 2), packing questions and related problems about the transition from symmetric to non-symmetric convex bodies (see Section 3). Furthermore, we investigated ball polyhedra, spindle convexity, and related notions (see Section 4), and a mostly combinatorial problem on separating a set, that is, the identifiability of any element of the set by knowing which sets of a given family contain it (see Section 5).

The topics discussed in Sections 1 and 3 cover the core important parts of the project proposal. Section 4 discusses results that are closely related to the problems on reduced convex bodies, mentioned in the project proposal. In Section 2, we describe a result that goes beyond the original proposal, and is a good starting point for us to widen the scope of our future research.

1. Covering a Convex Body

This section describes the results in the articles [6, 7] and [2].

A central question of the project was the *Levi–Gohberg–Markus–Boltyanski–Hadwiger Covering Conjecture* (or Illumination Conjecture). One way to state the problem goes as follows: We call the minimum number of smaller positive homothetic copies of a convex body K in real n-space needed to cover K the *illumination number* of K. Prove that it is at most 2^n, and that the bound is only attained by affine images of the cube.

We took two different approaches to studying this problem.

First, in [6], We constructed a convex body which is very close to the d-dimensional Euclidean ball, and yet, is of exponentially large illumination number. More specifically, we proved

Theorem 1.1. *Let $1 < D < 1.116$ be given. Then for any sufficiently large dimension n, there is an o-symmetric convex body K in $\mathbb{R}^n$, with illumination* *number at least* $.05D^n$*, for which* $\frac{1}{D}\mathbf{B}^n \subset K \subset \mathbf{B}^n$*, where* $\mathbf{B}^n$ *denotes the Euclidean unit ball.*

The illumination number is well known to be non-continuous on the space of convex sets in a fixed dimension n. Our result shows how strongly it fails to be continuous. Moreover, it shows that the gap between the *illumination parameter*, introduced by K. Bezdek [17], and the *vertex index*, introduced by K. Bezdek and Litvak [16], is very large for certain convex bodies.

The method of [6] is novel in the area, this is the first result where the covering number is bounded from below using a probabilistic construction. This paper is cited in [42, 18].

Second, in [7], we considered the opposite direction: finding upper bounds for the number $N(K, L)$ of translates of a given convex set L in $\mathbb{R}^n$ needed to cover another given convex set K. Finding $N(K, L)$ is an integer programming problem. Its linear relaxation (the corresponding LP problem) is obtained by considering the *fractional covering number*, $N^*(K, L)$, introduced in [35, 13].

Using a probabilistic argument, Artstein–Avidan and Slomka [13] found a bound of $N(K, L)$ in terms of $N^*(K', L')$, where K' and L' are almost K and L, respectively. A somewhat stronger bound was obtained in [7]. The main strength and novelty of [7] is that it gives a very simple, non-probabilistic proof. It is based on a combinatorial lemma of Lovász [34] and Stein [41]. With this method, we also obtained a simple proof of the fundamental result [38] of Rogers on economically covering $\mathbb{R}^n$ by translates of a convex body. Our method is the first non-probabilistic argument that has been used to obtain such general upper bounds for covering problems. Our method has been employed by Prosanov in [37] to obtain a simple proof of a bound on the chromatic number of $\mathbb{R}^n$ by Larman and Rogers [33].

As an invited contribution for the collection 'New Trends in Intuitive Geometry', in [2], we surveyed classical and recent results and methods on translative covering problems.

2. Proof of a conjecture of Bárány, Katchalski and Pach

Here, we describe the paper [8].

In [14], Bárány, Katchalski and Pach proved the following *quantitative* variant of the classical *Helly theorem*: If the intersection of a family of convex sets in $\mathbb{R}^n$ is of volume one, then the intersection of some subfamily of at most $2n$ members is of volume at most some constant $v(n)$. In [15], the bound $v(n) \leq n^{2n^2}$ is proved and $v(n) \leq n^{cn}$ is conjectured, where $c > 0$ is a universal constant. It has been confirmed in [8].

Theorem 2.1. *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^n$ such that the volume of its intersection is $vol\,(\cap\mathcal{F}) > 0$. Then there is a subfamily $\mathcal{G}$ of $\mathcal{F}$ with $card\,(\mathcal{F}) \leq 2n$ and $vol\,(\cap\mathcal{G}) \leq e^{n+1}n^{2n+\frac{1}{2}}vol\,(\cap\mathcal{F})$.*

The order of magnitude n^{cn} in the Theorem (and in the conjecture in [15]) is sharp as shown in [8]. The exponent, $(2n + \frac{1}{2})$, however can be improved on. Based on the method in [8], it was improved to $3n/2$ in [21].

It was cited in [21, 24, 12, 39, 23, 40, 22, 31, 19]. The study of quantitative Helly type questions is in the forefront of current research in discrete geometry, as can be gauged from the number of papers that have been published recently.

The result may be interpreted both as a Helly type result on intersection patterns of convex sets, as well as in the context of approximation of convex bodies by polyhedral sets.

3. Packing questions and non-symmetric convex sets

In [1] and [3], we studied *Minkowski arrangements*, that is, families of homothetic copies of a fixed centrally symmetric convex body, where no homothet contains the center of another, motivated by works of L. Fejes Tóth [26, 25] and Füredi and Loeb [28] (see also p. 133 of [20]). The problem of bounding the maximum size of a Minkowski arrangement is a close relative of the Bezdek–Pach conjecture, where pairwise touching homothets of a convex body are considered.

We gave upper and lower bounds for the maximum size of such a family, and extended the definition (and bounds) to non-symmetric convex sets as well. In doing so, we developed tools that may be used in approaching problems in discrete geometry concerning non-symmetric convex bodies.

Our paper [1] is cited, and some results are developed further in [36].

4. Intersections of balls

In [10, 9, 11, 4], we studied questions related to intersections of balls.

A *ball-polyhedron* in Euclidean n-space is a set with non-empty interior obtained as an intersection of finitely many unit balls. One milestone in their use is the proof of the finite case of Borsuk's conjecture in dimension three, independently by Grünbaum [29] and Heppes [30].

Cauchy's rigidity theorem states that a three–dimensional convex polyhedron is determined by its face lattice and its faces, up to a rigid motion. In [10], continuing previous research, carried out partly by the two authors, partly by others, we investigated rigidity properties of ball polyhedra.

In [11], we studied how *spindle convexity*, a natural notion in the study of ball polyhedra, extends to normed spaces. We also studied spaces where the unit ball is not centrally symmetric. The paper is cited in [27, 32]. Continuing these investigations, in [4], we considered a classical *question of Borsuk's* on partitioning sets into sets of smaller diameter. We gave bounds for the number of parts needed in various normed spaces.

In [9], we extended a classical Helly type result, Krasnosselsky's theorem to the notion of spindle convexity; it is cited in [32].

5. Separation with restricted families of sets

In [5] we first studied the following combinatorial problems.

Given a finite n-element set X, a family of subsets $\mathcal{F} \subset 2^X$ is said to *separate* X if any two elements of X are separated by at least one member of $\mathcal{F}$. We showed that if $|\mathcal{F}| > 2^{n-1}$, then one can select $\lceil \log n \rceil + 1$ members of $\mathcal{F}$ that separate X. If $|\mathcal{F}| \geq \alpha 2^n$ for some $0 < \alpha < 1/2$, then $\log n + O(\log \frac{1}{\alpha} \log\log \frac{1}{\alpha})$ members of $\mathcal{F}$ are always sufficient to separate all pairs of elements of X that are separated by some member of $\mathcal{F}$. This result is generalized to simultaneous separation in several sets. Analogous questions on separation by families of bounded Vapnik–Chervonenkis dimension are also discussed.

Finally, we considered separation of point sets in $\mathbb{R}^d$ by convex sets.

Publications of the project

[1] Márton Naszódi, János Pach, and Konrad Swanepoel. "Arrangements of homothets of a convex body". In: *Submitted to journal* (2017). URL: https://arxiv.org/abs/1608.04639.

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