Final report

on NKFIH grant 10461

Mechanical equilibrium and abrasion: a geometrical approach

1. Summary

The main result of this project is that we have established curvature-driven partial differential equations, in particular, Bloore's equation, as credible and reliable models for the shape evolution of sedimentary particles. We showed that natural fragments, serving as the initial conditions of this process display universal geometric features and we also showed that curvature-driven abrasion has two, well defined geometric phases. These findings provided powerful tools to analyze various natural scenarios. We able to determine crucial characteristics of fluvial abrasion in two geological environments, in particular, we could show definitive results on the much debated question about the relationship between sorting and abrasion in fluvial environments. We could also successfully deduce the provenance of Martian pebbles, relying solely on the pictures of NASA's Curiosity rover. Our work yielded a close estimate on the original size of thos pebbles from which their travel distance could be deduced. The latter serves as new evidence for past fluvial activity on the Red Planet. Using our mathematical models, we also provided explanation for the polyhedral shape of sand-abraded desert rocks, the so-called ventifacts. The model for aeolian abrasion can be regarded as a special limit of the Bloore equation where curvature-dependence vanishes and instead of smoothing, the evolution pushes towards singularities which are clearly visible as edges and vertices appearing on desert rocks. Our model also edmitted the study of upward-facing riverbed profiles, this corresponds to yet another special case of the Bloore equation where instead of radially uniform distribution, abraders arrive on approximately parallel trajectories. We showed that this case leads to the spontaneous emergence of circular profiles, which could be verified not only by computer simulations, but also matched existing laboratory data very closely.

One special focus of our research was the connection between mechanical equilibria and geometric shape. While this is a classical subject, little is known about how the number of equilibrium points may evolve under various geometric partial differential equations, in particular, curvature-driven flows. Similarly, not much is known about how truncation would influence this number. In this project we proved that the above-mentioned curvature-driven models, in particular, Bloore's equation reduce the expected value of the number N of mechanical equilibria (if the latter is defined as a random variable). Apparently, this is a delicate property of the model based on which its physical significance can be verified. Classification schemes are the basis to understand the evolution of equilibria. Earlier we have established the so-called primary classification system for convex bodies, based on the number N and stability type of their mechanical equilibria and proved the so-called Columbus algorithm which guarantees that new equilibria may be added to an arbitrary, generic convex body by suitable small truncations. Now we showed why an inverse algorithm (reducing N on an arbitrary convex body via any pre-defined truncations) might be hard to find. We also introduced a more refined, secondary classification, which is based on the Morse-Smale complex of the body and thus carries information not only on the number and stability type, but also on the topological arrangement of equilibrium points. We proved the completeness of the secondary classification scheme. This scheme appears to play a key role in the description of natural abrasion processes.

2. Results

In this report we will quote verbatim the research goals from the proposal and after each goal we summarize the *relevant results* and list the *main corresponding publications*.

2.1. Classification and description of convex bodies based on static equilibria

Our goal is to construct a classification scheme based on the Morse-Smale complex associated with the convex body. This secondary scheme would be refined version of the earlier (primary) classification based on the number and type of equilibrium points. Since already the primary (coarse) classification scheme proved to be useful from the geological point of view, we hope to get even more relevant information from the secondary classification. We are also interested in other aspects of the geometrical arrangement and number of equilibrium points, in particular, the numbers and geometry of imaginary points defined in [Domokos, Langi and Szabó, 2011]. We address the following questions and research goals:

2.1.1 Is the secondary classification complete in the sense that there is no empty secondary class? The completeness of the primary classification is known from earlier results.

Our aim is to prove the completeness of the secondary classification system.

In connection with 2.1.1, in [1] we proved that using the Columbus algorithm, the secondary classification system is incomplete in the sense that only a finite number of secondary classes can be reached if one starts from the Gömböc. We also identified other "ancestor" shapes under the Columbus algorithm.

In [2] we proved the completeness for the secondary classification by generalizing the Columbus algorithm via results from Morse theory, graph theory and convex geometry. We showed that combinatorial operations known as vertex splittings can be always realized on Mores-Smale complexes associated with convex bodies by using suitable local truncation. Our proof also indicates that the secondary classification system is extremely rich.

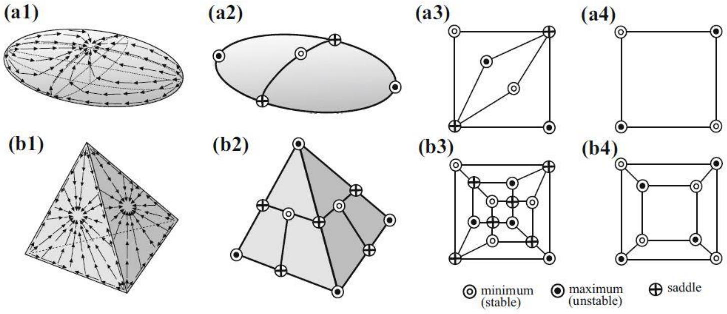

Figure 1. *Examples for primary and secondary equilibrium classes [2]. (a1-b1) Gradient fields on the ellipsoid and the tetrahedron (a2-b2) Morse-Smale complexes (a3-b3) Planar drawing of the graph representing the secondary class (a4-b4) quasi-dual representation of the complex*

[1] Kápolnai R., Domokos G., Szabó T.: **Generating spherical multiquadrangulations by restricted vertex splittings and the reducibility of equilibrium classes**. *Periodica Polytechnica Elelectrical Engineering*, Vol 56, No 1 (2012), pp. 11-20

[2] Domokos G., Lángi, Z., Szabó T.: **A topological classication of convex bodies**. *Geometriae Dedicata*, Vol 182, Issue 1 (2016), pp 95–116, DOI:10.1007/s10711-015-0130-4

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2.1.2 Is there a connection between the secondary class of pebble shapes, their geometric shape and the statistical frequency they appear in natural environments? To learn about these questions we need to determine the secondary class of statistically meaningful samples from various locations. We are also interested in the numbers of imaginary equilibrium points. To this end our proposal contains the budget for a high-precision laser scanner.

Our aim is to establish relevance of the secondary classification system from the point of view of natural sciences. We plan to determine the secondary classes for pebbles. This work will involve field trips to collect pebbles in relevant geological environments, laboratory work to obtain 3D images from the collected pebbles and numerical work to determine the secondary classes based on the images. Our work also includes the investigation of imaginary equilibria on pebble surfaces.

2.1.3 Can we track abrasion processes by using traditional or equilibrium-based classifications? Is the path in these classification systems characteristic for the physical parameters of the abrasion process? To this end our proposal contains the budget for a tumbler machine which we plan to modify for the purpose of these experiments.

Our aim is to determine the relevance of traditional and equilibrium-based classifications from the point of view of abrasion processes. This project will involve field work as well as laboratory experiments in a tumbler machine which we will modify to accommodate these experiments. It also involves the design and implementation of numerical algorithms capable to track abrasion processes of individual pebbles. In particular, we plan to extend the 2D 'chipping algorithm' we introduced in [Sipos et al 2011] to the full 3D process.

In connection with 2.1.3, we participated in a field work in South-Western Australia. Along the Williams river we collected over 1.500 pebbles at 12 locations. Using our "box model" [12] we fitted the model parameters to the field data and found a remarkable match between the two [3]. Along the river, we also measured the evolution of the number of equilibrium points, however, this part of the work is not yet published.

Figure 2*: Left: 12 sampling locations along the Williams river [3]. Right: Typical view of the river.*

We proved [4] that if the number of static equilibrium points is decreasing (e.g. in an abrasion process) then the transition from equilibrium classes {2,i} to {1,i} (i=1,2,…) and from classes from {i,2} to {i,1} (i=1,2,…) is much more difficult than other transitions. This mathematical result may be of key importance once the general trend of equilibria under natural abrasion can be established. We also showed [5] that if we average the number of equilibrium points over all internal points as reference points then this average does, in general not decrease under curvature-driven abrasion. In [6], based on tumbler experiments and the 3D 'chipping algorithm', we showed that curvature-driven abrasion naturally leads to a two-phase model of shape evolution. While we did not prove anything for the general trend of equilibria under natural abrasion, in the examples we examined (both numerically and experimentally) we found that the number of equilibrium points is decreasing.

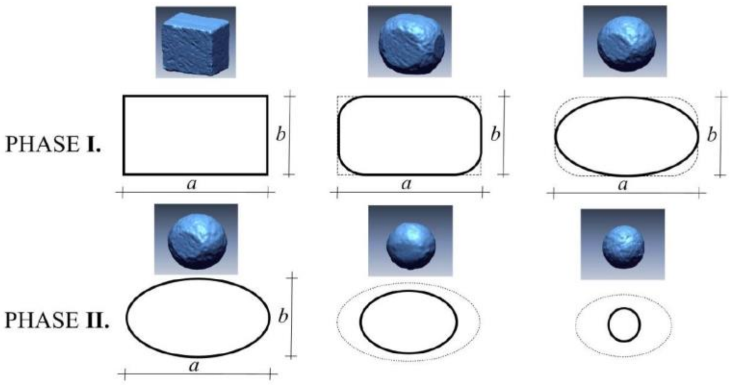

Figure 3*: Conceptual plot from [6] for two-phase curvature-driven abrasion. Blue insets: 3D scans from laboratory experiments. Observe initial sharp increase of isoperimetric ratio at approximately constant axis ratio in Phase I.*

We participated in a field work in Puerto Rico along the Bisley-Mameyes river system [7]. Our US partner, the group led by D.J. Jerolmack had a running grant to investigate sedimentary transport in this river system. Using their expertise, by joining the group we could identify a decreasing trend in the number of equilibrium points. Using our "box model" [12] we fitted the model parameters to the field data and found a remarkable match between the two data sets.

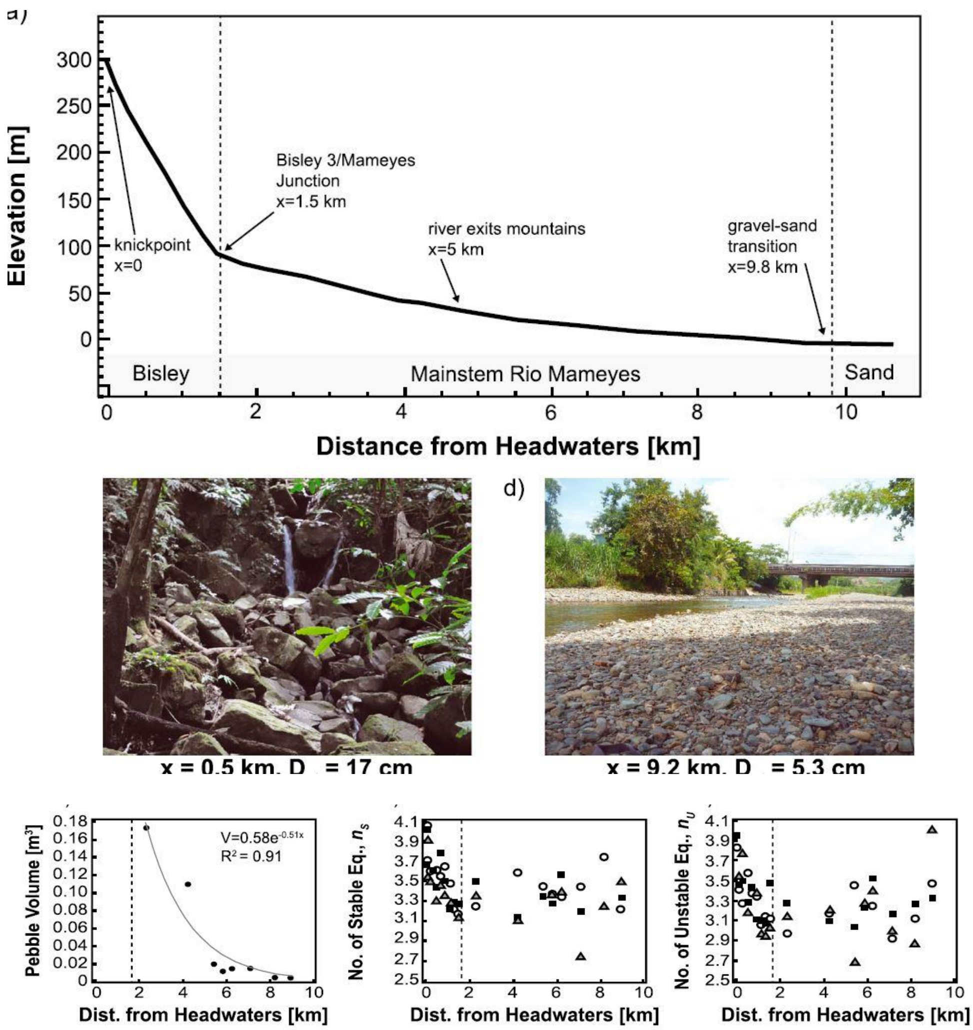

Figure 3: The Bisley-Mameyes system [7]. (a) Longitudinal section. (b) Pictures close to the source and to the lower end. (c) Field data on volume and number of equilibria.

Also in connection with 2.1.3, we proved [8] that under some physically plausible assumptions (e.g. that bifurcations are generic and the motion of the center of gravity can be modeled as white noise) the evolution of the number of critical points under a broad class of curvature-driven PDEs (which includes the Bloore equation) can be modeled as a random variable with monotonically decreasing expected value. This statement is of key importance for geophysical abrasion because it demonstrates that curvature-driven PDEs are likely the correct models for these processes. We also showed [9] that the geometric features of fragments, which serve as initial conditions of these abrasion processes, show universal characteristics.

Utilizing the mathematical and physical ideas from [8] and [9] we were able [10] to track the geological history of pebbles on Mars, using images from NASA's Curiosity rover. Our findings serve as one of the most compelling evidences of extended fluvial activity in the past on the Red Planet.

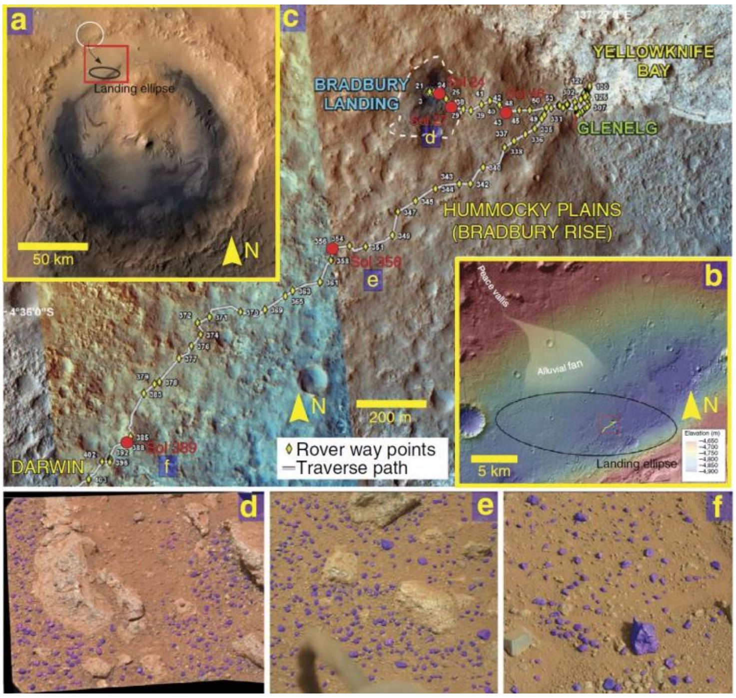

***Figure 4**: Mars field setting and the traverse of Curiosity [10]. (a) Gale crater (b) Curiosity's traverse for Sols 0-403 (yellow line) and Sols 403-817 (black line). (c) Expanded view of traverse on Sols 0-403. (d) Rounded pebbles at Sol 27 (e) Rounded pebbles at Sol 356 (f) Angular clasts at Sol 389.*

[3] Szabó T., Fityus S. Domokos G.: **Abrasion model of downstream changes in grain shape and size along the Williams River, Australia**. *Journal of Geophysical Research: Earth Surface* Vol 118, Issue 4 (2013), pp. 2059–2071

[4] Domokos G., Lángi Z.: **The robustness of convex solids**. *Mathematika* Vol 60 (2014), pp. 237-256, DOI:10.1112/S0025579313000181

[5] Domokos G., Lángi Z.: **On the average number of normals through points of a convex body**. *Stud. Sci. Math. Hung* Vol 52 No.4 (2015), pp. 434 - 449, DOI: 10.1556/012.2015.52.4.131

[6] Domokos G., Jerolmack D.J., Sipos A.Á., Török, Á.: **How river rocks round: resolving the shape-size paradox**. *PloS One Vol 9(2)*, (2014), Paper e88657, DOI: 10.1371/journal.pone.0088657

[7] Miller K.L., Szabó T., Domokos G., Jerolmack D.J.: **Quantifying the significance of abrasion and selective transport for downstream fluvial grain size evolution**. *Journal of Geophysical Research/ Earth Surface* Vol 119, Issue 11 November 2014, pp. 2412–2429, DOI: 10.1002/2014JF003156

[8] Domokos G.: **Monotonicity of spatial critical points evolving under curvature-driven flows.** *Journal of Nonlinear Science* Vol 25, Issue 2 (2015), pp 247-275, DOI 10.1007/s00332-014-9228-3

[9] Domokos G., Kun F., Sipos A.Á., Szabó T.: **The universality of fragment shapes.** *Nature Scientific Reports* 5, (2015), Article number: 9147, DOI:10.1038/srep09147

[10] Szabó T., Domokos G., Grotzinger, J.P., Jerolmack, D.J.: **Reconstructing the transport history of pebbles on Mars.** *Nature Communications* 6 (2015), Article number: 8366, DOI:10.1038/ncomms9366

2.2 Stationary solutions in abrasion processes for 2D convex shapes in Bloore's equations

2.2.1 In OTKA 72146 we found [Sipos et al. 2011] stationary solutions for unilaterally abraded bedrock profiles. Can this phenomenon be explained from Bloore's equations for the special case of unilateral abrasion?

Our aim is to establish the existence of stationary, travelling-wave solutions in the special, 2D case of unilateral abrasion as described by the special Bloore equations. We plan to compare these profiles with numerical simulations as well as laboratory experiments.

We formulated Bloore's equation for the special case of unilateral abrasion and proved that in the PDE travelling-wave like solutions exist. We identified these stationary solutions as circular arcs. We compared the analytical result to numerical simulation as well as laboratory experimental data and found a remarkable agreement. [11]

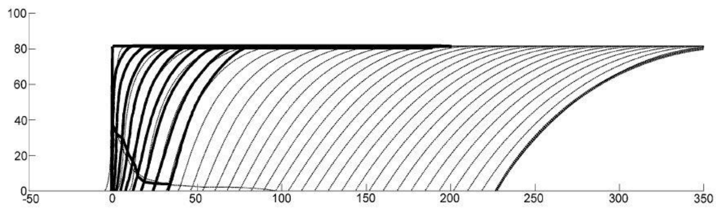

***Figure 5**: Bedrock profile evolution [11]. Bold line: experiments, dashed line: numerical model simulation. Observe as profile approaches steady-state circular geometry.*

[11] Domokos G. Gibbons G.W., Sipos A.Á: **Circular, stationary profiles emerging in unidirectional abrasion** *Mathematical Geosciences*, Vol 46 (2014), pp. 483-491

2.3 Self-similar solutions in abrasion processes for 3D convex shapes in Bloore's equations

2.3.1 Bloore' equation describes the abrasion of a single pebble in a given environment.

Our goal is to obtain a meaningful statistical description for the abrasion of pebble populations, based on Bloore's equation.

2.3.2 Does Bloore's collisional PDE admit homothetic solutions for abrading convex bodies? We seek the answer either in the full equation or, if this is not accessible, in a physically meaningful simplification

Our aim is to establish the existence or non-existence of attracting homothetic solutions in Bloore's PDE or some physically meaningful approximation thereof. We seek a *dynamical theory of mutual abrasion* in the spirit of dynamical systems theory and population dynamics, as opposed to the existing geometric PDE models of individual abrasion. We are interested in the numerical simulation of this process and its comparison to laboratory experiments.

2.3.3 Is the existence or non-existence of homothetic solutions connected with geological segregation?

Our aim is to establish the connection between the two most important geological observations about pebbles: the existence of dominant pebble ratios and the segregation of pebbles by size.

2.3.4 Can the statistical stability of such solutions be formulated and verified?

Our goal is to obtain an understanding how the dynamcal stability of homothetic solutions in a deterministic system could be related to their stability in a stochastic process.

We constructed an approximation of Bloore's PDE model, based on a 3-degree-of-freedom model of Ordinary Differential Equations (ODEs), corresponding to the evolution of the axes of ellipsoids. This (heuristic) approximation (which we called the 'box model'), not only admits to study the global dynamics but also serves as the basis of statistical simulation of collective abrasion. The box model can also be readily extended by frictional terms and we showed that in the presence of the latter attractors appear. The statistical stability of these attractors was tested in the stochastic simulation and we found that ongoing segregation by size can stabilize these attractors. We summarized our results in [12].

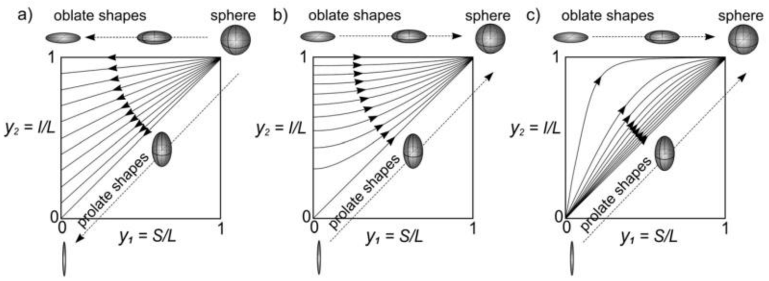

Figure 6*:Effect of the three terms in the box model of the Bloore equation on an ellipsoid with axes L>I>S. (a) Eikonal (constant) term (b) Mean Curvature term (c) Gaussian Curvature term*

[12] Domokos G, Gibbons G.W.: **The evolution of pebble shape and size in space and time**. *Proceedings of the Royal Society London, Series A.* Vol 468 no. 2146 (2012) pp. 3059-3079, DOI:10.1098/rspa.2011.0562 1471-2946

We also investigated the special case of the 3D Bloore equation where all curvature-driven term vanish, also known as the Eikonal flow. Under this equation one can prove the dynamical stability of a few limit shapes which we could clearly identify in Nature, in particular among sand-blasted rocks [13].

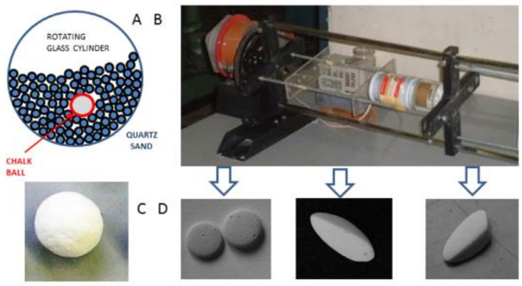

Figure 7: Scheme (A) and photo (B) of the experiment modeling Eikonal abrasion [13]. Initial (C) and final shapes (D) of experimental chalk particles. Latter match well theoretical predictions for stable limit shapes.

[13] Várkonyi, P.L., Laity J., Domokos G.: **Quantitative modeling of facet development in ventifacts by wind abrasion**. *Aeolian Research*, Vol 20 (2016), pp. 25-33, DOI: 10.1016/j.aeolia.2015.10.006

2.4 Other applications of the geometric approach to surface evolution problems

We plan to apply the tools developed in the frame of this project to a broader range of surface evolution problems including applications in elasticity theory and structural mechanics.

We gave a space-time interpretation of torsion in elastic bodies based on a geometric surface evolution model. We summarized our results in [14].

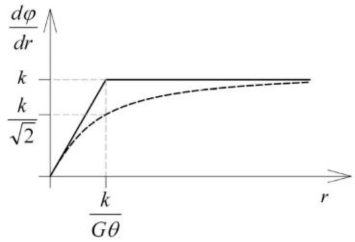

Figure 8: Comparison between the models. Solid line: Prandtl-Nadai theory. Dashed line: Our model based on Born-Infeld theory. Maximal deviation at elasto-plastic limit, common limit both for function value and first derivative at r=0 and r=∞.

[14] Domokos, G., Gibbons, G.W.: **Spacetime interpretation of torsion in prismatic bodies**. *Journal of Elasticity*, Vol 110, Issue 1 (2013), pp. 111-116