# Final report on the OTKA project *Error estimations for discontinuous Galerkin methods: a conforming approach*

## Introduction: a short explanation and preliminary convergence results for discontinuous Galerkin methods

Discontinuous Galerkin (dG) methods provide a powerful tool to approximate the solution $u$ of several kinds of partial differential equations. We discuss here their use for elliptic boundary value problems.

In short, the main idea of the dG methods is that the approximation is constructed as a discontinuous function. These are defined piecewise such that values on neighboring subdomains are completely independent. This flexibility makes possible the straightforward application of adaptive solution strategies. It is however, non-trivial how to include the jumps on the interelement faces into the corresponding variational formulation. Depending on this, a variety of dG methods can be defined.

Following the pioneering paper [1] on the systematic analysis of dG methods for elliptic boundary value problems the method became extremely popular and applied to a wide family for real life problems. A few monographs [4], [6] have also been published on this topic focusing mostly on implementation issues, while also a solid theoretical background has been discussed in [3].

To support the understanding of the problems and the corresponding results we use the following simple notations (without full explanation and mathematical rigor):

- The problem to investigate:
$$\begin{cases} \Delta u = g & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$
- $a(u,v) := \int_\Omega \nabla u \cdot \nabla v$ – the bilinear form corresponding to the Laplacian with homogeneous Dirichlet boundary conditions.
- $V_{\mathrm{DG}}$ – a generic dG finite element subspace.
- $a_{\mathrm{DG}}(\cdot,\cdot)$ – a generic dG bilinear form.
- $a_{\mathrm{IP}}(\cdot,\cdot)$ – a symmetric *interior penalty* (IP) bilinear form.
- $u_{\mathrm{DG}}$ – a corresponding dG approximation of $u$ in (1).
- $\bar{v}_{\mathrm{DG}}$ – a smoothing of an element in $V_h$.
- $h$ – a mesh parameter; we assume here quasi-uniform non-degenerated mesh.
- $\eta_h : \mathbb{R}^d \to \mathbb{R}$ a constant function supported on the ball $B(\mathbf{0}, h^s)$ with the parameter $s \in \mathbf{R}^+$ such that $\int_{\mathbb{R}^d} \eta_h = 1$.
- $\nabla_h$ – the piecewise gradient.
- $[\![u]\!]$ and $\{\!\{u\}\!\}$ – jump and average operators defined on the interelement faces.
- $\mathbb{P}_{h,k}$ – locally polynomial (discontinuous) finite element space.

In one sentence, the overall aim of this study was to improve the *numerical analysis* of the existing dG methods. The main question in the analysis of any numerical method is the convergence, or in more details:

- In what sense does it converge?

- Are there special conditions for the convergence?
- What is the convergence speed?

In the present research, we tried to answer these questions in case of elliptic boundary value problems for a particular kind of dG methods.

The results in the literature were restricted to the $L_2$-norm and to the so-called dG norm given by $\|u_{\mathrm{DG}}\| = \sqrt{a_{\mathrm{DG}}(u,u)}$.

In real-life problems, however, the solution should be continuous, which gives rise to the post-processing of the discontinuous approximations.

## Results obtained in the project

In short, our main result is that we have verified that the local average of a certain kind of dG method delivers quasi-optimal error estimate in the $H^1$-seminorm.

The basic idea of the analysis is that in the bilinear form we are looking already for smoothed discontinuous approximations, i.e. as the solution of the following variational problem: Find $u_{\mathrm{DG}} \in V_h = \mathbb{P}_{h,k}$ such that

$$a(\bar{u}_{\mathrm{DG}}, \bar{v}_{\mathrm{DG}}) = (-f, \bar{v}_{\mathrm{DG}}) \quad \forall v_{\mathrm{DG}} \in V_h.$$

From one point of view, it is composed from discontinuous approximations. On the other hand, it makes possible a conforming error analysis if $\bar{u}_{\mathrm{DG}} \in H^1(\Omega)$.

According to this approach, the following main steps are to be executed:

- Rewrite the bilinear form $a(\bar{u}_{\mathrm{DG}}, \bar{v}_{\mathrm{DG}})$ in concrete terms, such that it becomes a function of $u_{\mathrm{DG}}$ on the subdomains, $[\![u_{\mathrm{DG}}]\!]$ and $\{\!\{u_{\mathrm{DG}}\}\!\}$ on the interelement faces and of these terms of $v_{\mathrm{DG}}$.
- Estimate the difference of the terms in the above expansion with the terms in a conventional dG bilinear form: We made a comparison with an overpenalized IP method.
- Give an upper bound for $|a_{\mathrm{IP}}(u_h, v_h) - a(\bar{u}_h, \bar{v}_h)|$.
- Establish the quasi optimal convergence of the average of an IP approximation.

**First year** In the first year of the project we investigated the one-dimensional case. It turned out that the error analysis can be performed using the averaging $\bar{u}_{\mathrm{DG}} = \eta_h * u_{\mathrm{DG}}$.

It took some time, to make this decision. We have rather tried to deal with distributions than deal with complicated classical terms, which were the consequence of the classical smoothing procedure using higher-order splines.

In concrete terms, we have proved that the bilinear form $a(\bar{u}_{\mathrm{DG}}, \bar{v}_{\mathrm{DG}})$ is a lower-order modification of $a_{\mathrm{IP,s}}(u,v)$, where $a_{\mathrm{IP,s}}$ is an overpenalized dG bilinear form, corresponding to a popular version of dG methods. Since we could perform an $H^1$-conforming error analysis, an important consequence could be drawn:

- ▶ The smoothing $\bar{u}_{\mathrm{DG}}$ of the overpenalized dG approximation converges to $u$ in a quasi optimal way *with respect to the energy seminorm.*

In this case, I could involve a PhD student Gábor Csörgő to help me in the numerical experiments. These confirmed that the bilinear forms $a(\bar{u}_{\mathrm{DG}}, \bar{v}_{\mathrm{DG}})$ and $a_{\mathrm{IP,s}}(u,v)$ are really very close to each other. One could hardly see their difference in the experiments.

These results have been published with full details in [2].

**Second year** For the multidimensional generalization we had to review some results of the distribution theory. By taking the gradient of a discontinuous piecewise polynomial function $u_h \in \mathbb{P}_{h,k}$ we obtain piecewise polynomials and non-regular distributions supported on the interelement faces:

$$\nabla u_h = \nabla_h u_h + [\![u_h]\!] \,,$$

where $[\![u_h]\!]$ denotes the non-regular component. This has a close connection with the jump function. With these we obviously have

$$\begin{aligned} a(\eta_h * u_h, \eta_h * v_h) &= (\eta_h * \nabla_h u_h, \eta_h * \nabla_h v_h) + (\eta_h * \nabla_h u_h, \eta_h * [\![v_h]\!]) \\ &\quad + (\eta_h * \nabla_h v_h, \eta_h * [\![u_h]\!]) + (\eta_h * [\![u_h]\!] \,, \eta_h * [\![v_h]\!]). \end{aligned} \tag{2}$$

The main difficulty in the construction of a more explicit form of (2) is the computation of the convolution $\eta_h * [\![u_h]\!]$. Surprisingly (which do not match with the one-dimensional case), it turns out that this will be a continuous function, which can be given in a closed form. In this way, we obtained an expansion of $a(\eta_h * u_h, \eta_h * v_h)$. We note also the following interesting observations:

- The Green formula implies a clear connection between lifted formulations and direct forms: the volume integrals $(\eta_h * \nabla_h u_h, \eta_h * [\![v_h]\!])$ can be expressed as surface integrals $(\eta_h * \eta_h * \nabla_h u_h, [\![v_h]\!])$.
- The so-called penalty terms could be computed explicitly, in the two-dimensional case we obtained $\frac{16}{3\pi^2} h^s([\![u]\!] \,, [\![v]\!])_f$, while in the three-dimensional case $\frac{3}{5} h^s([\![u]\!] \,, [\![v]\!])_f$ on a face $f$.
- The averaging parameter $s$ should satisfy $s > d + 2$, where $d$ denotes the space dimension.
- In the present analysis we did not need any smoothness assumption on the analytic solution.

We could finally compare the bilinear form $a(\eta_h * u_h, \eta_h * v_h)$ with $a_{\mathrm{IP,s}}(u_h, v_h)$, an overpenalized interior penalty bilinear form.

The systematic comparison of the terms in $a(\eta_h * u_h, \eta_h * v_h)$ and $a_{\mathrm{IP,s}}(u_h, v_h)$ gives that their difference - in case of a quasi uniform mesh - can be bounded by $Ch^{s-1}\|\nabla(\eta_h * u)\|\|\nabla(\eta_h * v)\|$. The final result is similar to the one-dimensional case:

$$\|\nabla(u - \eta_h * u_{\mathrm{IP},s})\| \lesssim \inf_{v_h \in \mathbb{P}_{h,k}} \|u - \eta_h * v\|_1 + \mathcal{O}(h^{s-\frac{1}{2}}) + h^d \|\eta_h * g - g\|, \tag{3}$$

which provides an estimate of the error between the analytic solution and the averaged dG approximation in the energy seminorm. The results of the multidimensional study have been summarized in [5], which was submitted for publication.

# A short summary of the consequences and ideas for further research

The first main consequence of the above study is that we could establish the convergence of a postprocessed dG approximation in the real energy (semi)norm.

The second main consequence is the possibility of an alternative introduction of dG methods:

- Use the bilinear form $a(\eta_h * u_h, \eta_h * v_h)$ with the averaged approximations.
- Simplify it using the approximation of the terms in (2) to obtain $a_{\mathrm{IP,s}}(u_h, v_h)$.

We stress here that the aim of this research was not to obtain a completely new numerical method. We have rather tried to explain why the existing methods (especially the symmetric interior penalty method) are performing well and in which sense do they really converge.

The research should be continued along the following lines:

- The most important step to make the error estimate in (3) more explicit is to give an upper bound for the term $\inf_{v_h \in \mathbb{P}_{h,k}} \|u - \eta_h * v\|_1$ with a given $u \in H^1(\Omega)$.
- Using this idea one could develop a posteriori error analysis for piecewise constant approximations.

# References


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