FINAL REPORT FOR OTKA GRANT PROPOSAL NO. 101855: AUTOMORPHIC FORMS AND L-FUNCTIONS

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We have made progress in three different, but interrelated topics: bounds for automorphic L-functions, bounds for automorphic forms, and bounds for prime gaps. In the first two topics we collaborated with Valentin Blomer (Göttingen), Péter Maga (Budapest), and Djordje Milićević (Bryn Mawr), while in the third topic we collaborated with the Polymath8 team led by Terence Tao. The results appear in the research papers [BH14, BHM16, BHMM, Po14a, Po14b] and in the surveys [Ha14a, Ha14b]. We have also contributed to a physics project [WLITH15] and a retrospective article [Po14c] on the Polymath8 project. A detailed account of the results is provided below.

1. Bounds for automorphic L-functions

The first result within the project is an extension of our Burgess-like subconvexity bound in [BH08] to cusp forms of arbitrary nebentypus.

Theorem 1 ([BH14])**.** *Let f be a primitive (holomorphic or Maaß) cusp form of archimedean parameter μ, level N and arbitrary nebentypus, and let χ be a primitive character modulo q. Then for $\Re s = 1/2$ and for any $\varepsilon > 0$ the twisted L-function satisfies*

$$L(f\otimes\chi,s) \ll_\varepsilon \left(|s|^{\frac14}\,|\mu|^{\frac12}N^{\frac14}(N,q)^{\frac18}q^{\frac38} + |s|^{\frac12}\,|\mu|\,N^{\frac12}(N,q)^{\frac14}q^{\frac14}\right)(|s|\,|\mu|\,Nq)^\varepsilon$$

if f is holomorphic, and

$$L(f\otimes\chi,s) \ll_\varepsilon \left(|s|^{\frac14}\,(1+|\mu|)^3N^{\frac14}(N,q)^{\frac18}q^{\frac38} + |s|^{\frac12}\,(1+|\mu|)^{\frac72}N^{\frac12}(N,q)^{\frac14}q^{\frac14}\right)(|s|\,(1+|\mu|)Nq)^\varepsilon$$

otherwise.

In combination with the convexity bound, this yields the clean inequality

$$L(f\otimes\chi,s) \ll_\varepsilon (|s|\,(1+|\mu|)Nq)^\varepsilon\,|s|^{\frac12}\,(1+|\mu|)^3N^{\frac12}q^{\frac38}.$$

Another corollary is the hybrid subconvexity bound

$$L(f\otimes\chi,s) \ll_{\mu,\varepsilon} (N|s|q)^\varepsilon N^{\frac45}(|s|q)^{\frac12-\frac1{40}},$$

but this has been superseded, at least in the $(|s|q)$-aspect, by Munshi [Mu14] and Wu [Wu14].

2. Bounds for automorphic forms

Our second result provides good bounds for the sup-norm of certain automorphic forms on the hyperbolic 3-space $\mathscr{H}^3$, extending the earlier results on the hyperbolic plane $\mathscr{H}^2$. For the latter we recall that the matrix group $\mathrm{GL}_2(\mathbb{R})$ acts transitively by hyperbolic isometries on $\mathscr{H}^2$, which leads to the identification $\mathscr{H}^2 \cong \mathrm{Z}(\mathbb{R})\backslash\mathrm{GL}_2(\mathbb{R})/\mathrm{O}_2(\mathbb{R})$. Then, concerning the Hecke congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ and the Laplace operator Δ on $\mathscr{H}^2$, Templier [Te15] proved the following state-of-the-art result.

Theorem 2 ([Te15])**.** *Let $N\in\mathbb{Z}$ be square-free, and let $\phi:\Gamma_0(N)\backslash\mathscr{H}^2\to\mathbb{C}$ be an L^2-normalized Hecke–Maass cusp form. If $\Delta\phi=\lambda\phi$, then $\|\phi\|_\infty \ll_\varepsilon (\lambda|N|)^\varepsilon\lambda^{\frac5{24}}|N|^{\frac13}$.*

Here and later, L^2-normalized is always meant with respect to the invariant probability measure. By comparison, the trivial (or easy) bound would be $\|\phi\|_\infty \ll_\varepsilon (\lambda|N|)^\varepsilon \lambda^{\frac{1}{4}}|N|^{\frac{1}{2}}$.

Analogously, $\mathrm{GL}_2(\mathbb{C})$ acts transitively by hyperbolic isometries on $\mathscr{H}^3$, which leads to the identification $\mathscr{H}^3 \cong \mathrm{Z}(\mathbb{C})\backslash\mathrm{GL}_2(\mathbb{C})/\mathrm{U}_2(\mathbb{C})$. Then, concerning the Hecke congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z}[i])$ and the Laplace operator Δ on $\mathscr{H}^3$, we can state our result as follows.

Theorem 3 ([BHM16])**.** *Let $N \in \mathbb{Z}[i]$ be square-free, and let $\phi : \Gamma_0(N)\backslash\mathscr{H}^3 \to \mathbb{C}$ be an L^2-normalized Hecke–Maass cusp form. If $\Delta\phi = \lambda\phi$, then $\|\phi\|_\infty \ll_\varepsilon (\lambda|N|)^\varepsilon \min(\lambda^{\frac{5}{12}}|N|, \lambda^{\frac{1}{2}}|N|^{\frac{2}{3}})$.*

Here the trivial (or easy) bound reads $\|\phi\|_\infty \ll_\varepsilon (\lambda|N|)^\varepsilon \lambda^{\frac{1}{2}}|N|$. A nice feature of this theorem is that, separately in the eigenvalue and the level aspects, it is as strong as Theorem 2 for the surface case. In particular, in the level aspect we have a Weyl-type saving on $\mathscr{H}^2$ and $\mathscr{H}^3$, exactly as on the spheres $\mathscr{S}^2$ and $\mathscr{S}^3$ by the work of Blomer–Michel [BM13]. We note that in the hybrid aspect, Theorem 3 gives $\|\phi\|_\infty \ll_\varepsilon (\lambda|N|)^\varepsilon \lambda^{\frac{4}{9}}|N|^{\frac{8}{9}}$ by interpolation. It is also likely that the ideas of Saha [Sa14, Sa15] can be applied to remove the restriction that N is square-free, although the resulting exponents will be larger (but still non-trivial).

The proof of Theorem 3 involves delicate geometric and Diophantine arguments, in addition to the automorphic input. A key observation is that, identifying $\mathscr{H}^3$ with the half-space $\{z+rj : z \in \mathbb{C},\ r > 0\}$ inside the Euclidean space of quaternions $\mathbb{H} = \mathbb{R}+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$, the supremum of $|\phi(P)|$ is attained at a point $P = z+rj \in \mathscr{H}^3$ such that the lattice $\mathbb{Z}[i]+\mathbb{Z}[i]P \subset \mathbb{H}$ has favorable properties. For example, we can ensure that in any ball of radius R the number of lattice points is $\ll 1+R^2|N|+R^4r^{-2}$, where also $r \gg |N|^{-1}$. Very recently we managed to extend this crucial ingredient to Hecke–Maass forms over any number field in place of $\mathbb{Q}(i)$, hence it seems that the Weyl-type saving in the level aspect holds in general. This result will appear in our future work [BHMM].

3. Bounds for prime gaps

The Polymath8 project was initiated shortly after the sensational result of Zhang [Zh14] that a certain positive integer occurs as a prime gap infinitely often. Zhang's proof builds crucially on the earlier breakthrough of Goldston–Pintz–Yıldırım [GPY09] that established the existence of relatively very small prime gaps, which in turn relied on the classical work of Selberg [Se91] and Bombieri–Vinogradov [Bo65, Vi65]. In fact Zhang proved a stronger result.

Theorem 4 ([Zh14])**.** *There exists a positive integer k with the following property. If $\mathscr{H}$ is an admissible k-set, then for infinitely many positive integers n, the translated set $n+\mathscr{H}$ contains at least two primes.*

Here a k-set of integers is called *admissible* if it does not contain a complete system of residues modulo any integer $m \geqslant 2$. Theorem 4 should be compared with the famous

Dickson–Hardy–Littlewood Conjecture ([Di04, HL23])**.** *Let $\mathscr{H}$ be an admissible k-set. Then for infinitely many positive integers n, the translated set $n+\mathscr{H}$ consists of k primes.*

The main focus of the Polymath8 project was to establish Theorem 4 with k as small as possible, and to examine what it means for the smallest prime gap that occurs infinitely often. The first part of the project (cf. [Po14a]) improved and refined Zhang's original analysis, whereas the second part (cf. [Po14b]) focused on the more effective technique of Maynard [Ma15] and Tao [Ta13]. The table on the next page summarizes progress along these lines.

The proof of Theorem 4 relies on the existence, for any sufficiently large $x \geqslant 2$, of a probability measure on the integers $x \leqslant n \leqslant 2x$ for which the expected number of primes in $n+\mathscr{H}$ exceeds one. The probability measures that are known to work are inspired by the theory of the Selberg sieve [Se91], and the calculation of the relevant averages leads to the following hypothesis of Elliott–Halberstam [EH70].

source	$k=$	$\liminf(p_{n+1}-p_n)\leqslant$
Zhang [Zh14]	3.5×10^6	7×10^7
Polymath8a [Po14a]	632	4680
Maynard [Ma15]	105	600
Polymath8b [Po14b]	50	246

Hypothesis $EH(\theta)$**.** *For any constant* $A>0$ *we have*

$$\sum_{\substack{q\leqslant x^\theta \\ q\ \text{squarefree}}} \max_{(a,q)=1} \left| \sum_{\substack{x\leqslant p\leqslant 2x \\ p\equiv a \pmod q}} 1 - \frac{1}{\varphi(q)}\int_x^{2x}\frac{dt}{\log t}\right| \ll_A \frac{x}{\log^A x}.$$

This hypothesis holds for any $\theta<1/2$ by the work of Bombieri–Vinogradov [Bo65, Vi65], while Elliott–Halberstam [EH70] conjectured it for any $\theta<1$. The probability measure introduced by Maynard [Ma15] and Tao [Ta13] utilizes any k times continuously differentiable function $f:\mathbb{R}^k\to\mathbb{R}$ supported in the simplex $\Delta_k:=\{(t_1,\dots,t_k)\in\mathbb{R}^k:\ t_1,\dots,t_k\geqslant 0 \text{ and } t_1+\dots+t_k\leqslant 1\}$, and for this probability measure the expected number of primes in $n+\mathscr{H}$ equals, as x tends to infinity,

$$\frac{\theta}{2}\cdot\frac{k\int_{\mathbb{R}^{k-1}\times\{0\}}\left(\frac{\partial^{k-1}f}{\partial t_1\dots\partial t_{k-1}}\right)^2}{\int_{\mathbb{R}^k}\left(\frac{\partial^k f}{\partial t_1\dots\partial t_k}\right)^2}+o(1). \tag{1}$$

This reduces Theorem 4 to choosing θ and f in such a way that the product in (1) exceeds 1.

The probability measures used by Zhang [Zh14] and Polymath8a [Po14a] were restricted in the sense that they corresponded to auxiliary functions $f:\mathbb{R}^k\to\mathbb{R}$ of the form $f(t_1,\dots,t_k)=g(t_1+\dots+t_k)$. For such special f's and the available θ's, the product in (1) can get arbitrary close to 1, but it can never exceed it. Hence Zhang [Zh14] and Polymath8a [Po14a] focused on raising θ slightly above $1/2$ at the cost of weakening Hypothesis $EH(\theta)$. The weaker version of $EH(\theta)$ only concerns x^δ-smooth moduli q for a fixed $\delta>0$, and only those residue classes $a \bmod q$ whose reduction modulo any prime divisor $p\mid q$ equals one of the nonzero differences within the k-tuple $\mathscr{H}$. This idea goes back to Motohashi–Pintz [MP08]. In this way, Zhang [Zh14] could achieve any $\theta<1/2+1/584$, while in Polymath8a [Po14a] we could take any $\theta<1/2+7/300$. However, the restriction of the moduli q and the residue classes $a \bmod q$ also has a negative effect in (1), and this increases the admissible value of k compared to what would originally follow from (1). In Polymath8a [Po14a] we managed to decrease this negative effect substantially, partly by relaxing the restriction on q, and this culminated in the value $k=632$ and the existence of a prime gap at most 4680 that occurs infinitely often.

For the more general auxiliary functions $f:\mathbb{R}^k\to\mathbb{R}$, Maynard [Ma15] discovered that the second fraction in (1) can be as large as about $\log k$. That is, he proved a version of Theorem 4 in which $n+\mathscr{H}$ contains about $(\log k)/4$ primes infinitely often. In addition, he constructed an example $f(t_1,\dots,t_k)$ for $k=105$ that only depended on $t_1+\dots+t_k$ and $t_1^2+\dots+t_k^2$, yet it raised the product in (1) above 1. In Polymathb8 [Po14b] we refined and extended Maynard's work in various ways, e.g. by enlarging the support Δ_k of f slightly, or by constructing f in terms of more power sums $t_1^m+\dots+t_k^m$, or by combining the Maynard–Tao weights with the analysis of Zhang–Polymath8a. As a result, one can now take $k=50$ in Theorem 4, and even $k=3$ under a generalized Elliott–Halberstam conjecture. Moreover, for large k, the average number of primes in $n+\mathscr{H}$ can be raised from about $(\log k)/4$ to about $(157/600)(\log k)$.

4. A DETERMINANT INEQUALITY

Our final result is a determinant inequality for the split orthogonal group $O(n,n)$, which is useful in certain quantum Monte Carlo simulations. We recall that the group $O(n,n)$ consists of the $2n\times 2n$ real matrices M satisfying

$$M^T\eta M=\eta,\qquad \eta=\operatorname{diag}(\underbrace{1,\dots,1}_{n},\underbrace{-1,\dots,-1}_{n}),$$

and it has four connected components $O^{\pm\pm}(n,n)$ corresponding to the possible signs of the upper left and bottom right $n\times n$ subdeterminants of M (which are never zero). The inequality reads as follows.

Theorem 5 ([WLITH15])**.** *For $M\in O(n,n)$ we have*

$$\det(I+M)\begin{cases}\geqslant 0, & M\in O^{++}(n,n);\\ \leqslant 0, & M\in O^{--}(n,n);\\ =0, & \textit{otherwise}.\end{cases}$$

We record a case particularly important for physics.

Corollary ([WLITH15])**.** *Assume that the $n\times n$ real matrices A_j $(j=1,\dots,k)$ satisfy $\eta A_j\eta=-A_j^T$. Then we have*

$$\det\left(I+\prod_{j=1}^{k}e^{A_j}\right)\geqslant 0.$$

REFERENCES

[BH08] V. BLOMER, G. HARCOS, Hybrid bounds for twisted L-functions, *J. Reine Angew. Math.* **621** (2008), 53–79.

[BH14] V. BLOMER, G. HARCOS, Addendum: Hybrid bounds for twisted L-functions, *J. Reine Angew. Math.* **694** (2014), 241–244.

[BHM16] V. BLOMER, G. HARCOS, D. MILIĆEVIĆ, Bounds for eigenforms on arithmetic hyperbolic 3-manifolds, *Duke Math. J.*, to appear.

[BHMM] V. BLOMER, G. HARCOS, P. MAGA, D. MILIĆEVIĆ, On the sup-norm problem for GL_2 over number fields, in preparation.

[BM13] V. BLOMER, P. MICHEL, Hybrid bounds for automorphic forms on ellipsoids over number fields, *J. Inst. Math. Jussieu* **12** (2013), 727–758.

[Bo65] E. BOMBIERI, On the large sieve, *Mathematika* **12** (1965), 201–225.

[Di04] L. E. DICKSON, A new extension of Dirichlet's theorem on prime numbers, *Messenger of Math.* **33** (1904), 155–161.

[EH70] P. D. T. A. ELLIOTT, H. HALBERSTAM, A conjecture in prime number theory, In: *Symposia Mathematica*, Vol. IV (INDAM, Rome, 1968/69), 59–72, Academic Press, London, 1970.

[GPY09] D. A. GOLDSTON, J. PINTZ, C. Y. YILDIRIM, Primes in tuples I, *Ann. of Math. (2)* **170** (2009), 819–862.

[Ha14a] G. HARCOS, Twisted Hilbert modular L-functions and spectral theory, In: *Automorphic forms and L-functions* (ed. J. LIU), vol. 30 of *Adv. Lect. Math. (ALM)* 49–67, Int. Press, Somerville, MA, 2014.

[Ha14b] G. HARCOS, Prímek, Polignac, Polymath [Primes, Polignac, Polymath], *Mat. Lapok (N.S.)* **20** (2014), *no.* 2, 1–13. (Hungarian)

[HL23] G. H. HARDY, J. E. LITTLEWOOD, Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes, *Acta Math.* **44** (1923), 1–70.

[Ma15] J. MAYNARD, Small gaps between primes, *Ann. of Math. (2)*, **181** (2015), 383–413.

[MP08] Y. MOTOHASHI, J. PINTZ, A smoothed GPY sieve, *Bull. Lond. Math. Soc.* **40** (2008), 298–310.

[Mu14] R. MUNSHI, The circle method and bounds for L-functions - I, *Math. Ann.* **358** (2014), 389–401.

[Po14a] D. H. J. POLYMATH, New equidistribution estimates of Zhang type, *Algebra Number Theory* **8** (2014), 2067–2199; extended version available at arXiv:1402.0811v2

[Po14b] D. H. J. POLYMATH, Variants of the Selberg sieve, and bounded intervals containing many primes, *Res. Math. Sci.* **1** (2014), *no.* 12, 83 pp.

[Po14c] D. H. J. POLYMATH, The "Bounded gaps between primes" Polymath project - A retrospective analysis, *EMS Newsletter*, *no.* 94, December 2014, 13–23.

[Sa14] A. SAHA, On sup-norms of cusp forms of powerful level, preprint, arXiv:1404.3179

[Sa15] A. SAHA, Hybrid sup-norm bounds for Maass newforms of powerful level, preprint, arXiv:1509.07489

[Se91] A. SELBERG, Lectures on sieves, In: *Collected papers, Vol. II*, Springer-Verlag, Berlin, 1991.

[Ta13] T. TAO, Polymath8b: Bounded intervals with many primes, after Maynard, 2013, blog entry

[Te15] N. TEMPLIER, Hybrid sup-norm bounds for Hecke-Maass cusp forms, *J. Eur. Math. Soc. (JEMS)* **17** (2015), 2069–2082.

[Vi65] A. I. VINOGRADOV, The density hypothesis for Dirichet *L*-series (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 903–934; Correction (Russian), *ibid.* **30** (1966), 719–720.

[WLITH15] L. WANG, Y.-H. LIU, M. IAZZI, M. TROYER, G. HARCOS, Split orthogonal group: A guiding principle for sign-problem-free fermionic simulations, *Phys. Rev. Lett.* **115** (2015), *no.* 25, Art. ID 250601, 6 pp; extended version available at arXiv:1506.05349v3

[Wu14] H. WU, Burgess-like subconvex bounds for $\mathrm{GL}_2 \times \mathrm{GL}_1$, *Geom. Funct. Anal.* **24** (2014), 968–1036.

[Zh14] Y. ZHANG, Bounded gaps between primes, *Ann. of Math. (2)* **179** (2014), 1121–1174.