The modern theory of deterministic, optimal and random structures

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Beside me (M. Simonovits) Miklós Abért, Gábor Elek, Zoltán Füredi, Ervin Győri, Vera T. Sós, and Endre Szemerédi, participated in this project, and of course, our coauthors. In April 2015, with the permission of NKFI, Attila Sali also joined our project.

Large part of our work is a direct continuation of our earlier investigations. (Further, most probably, in the forthcoming years, we shall continue researching several topics mentioned below.) The most important subfields are perhaps

1. Turán type extremal graph and hypergraph problems.
2. Turán type extremal problems for other structures.
3. Connection between discrete and continuous theories, graph limits (dense and sparse theory).
4. Application of Regularity Lemmas in extremal graph theory.
5. Stability methods, and their applications to obtain sharp results.

There is no way to list here all those results proved by us which we feel important, I will restrict myself to a selection. Further, in the area of graph limits, understanding the results needs a lot of knowledge, so here I had to avoid going into details, primarily in the so called "sparse case".

Extremal problems. Extremal graph theory is today one of the most important and also most popular areas in Discrete Mathematics, and one

[0]Occasionally we used incorrectly the number 101535, however, the 101536 is the correct one.

which gave birth to several important methods in Discrete Mathematics, e.g., to Szemerédi Regularity Lemma, Blowup Lemma, Stability method, continuous methods in Combinatorics. Also, in the last few years an important new area evolved, based on the convergence of structures, above all, but not only, limits of dense and sparse graph sequences. It is strongly connected also to Ramsey Theory and Anti-Ramsey theory, and to some further graph colouring questions.

Reduction. For ordinary graphs there are many important results on **non-degenerate extremal problems**, (where non-degenerate means that the extremal graph sequences have positive edge density). Many of the problems can be reduced to **degenerate** extremal graph problems, where typically the extremal number is around $\sim n^{1+\alpha}$, for some $\alpha \in [0,1)$. (Such questions are, e.g., the extremal problems on complete bipartite graphs, even cycles, or when the excluded graph is a 3-dimensional cube.) We have several important results in these areas, among others on the cube and on the hexagon. And then come the **strongly degenerate** extremal problems, where the extremal function can be estimated by a linear function: these are the cases where there is a tree among the excluded graphs. I mention this classification, since large part of the research done in the frames of this project belongs to this later case, primarily to the case when path, trees, even cycles are excluded, or their hypergraph generalizations, and this area seems to be one of the most important areas in this field.

Stability Methods. There are several cases where the almost extremal structures are very similar to the extremal structures, and this phenomenon can be used to obtain absolutely sharp results, at least for large values of the parameters. These methods are used in proving the sharp Erdős-Sós conjecture, (Ajtai, Komlós, Simonovits, Szemerédi) and some new results connected to dual Anti-Ramsey theorems (Simonovits) and other stability results, (Füredi at al[1]).

Trees. Some of the work done in our group is directly connected to embedding trees into large graphs. This is centered around two results. One is the solution of the Erdős-Sós conjecture on the extremal number of trees, (Ajtai, Komlós, Simonovits, Szemerédi) the other is the approximate solution of

[1]Mostly I try to avoid using "at al", however here sometimes I mention explicitly only the names of the participants of the OTKA project, or some constant or very important coauthors...

the Loebl-Komlós-Sós conjecture asserting that if the **median degree** of a graph is k then it contains all the trees of this size (Hladký, Komlós, Piguet, Simonovits, Stein, Szemerédi).

The results assert that for sufficiently large trees these conjectures are true. The proofs use regularity lemmas and stability methods, have the nature that just to write them up takes an enormous amount of work. (One of the works (Loebl-Komlós-Sós conjecture) posted by us on the arXiv was more than 160 pages, later we decided to split this large work into four still quite long papers, which are also on the arXiv, and accepted for publication.) In case of these two conjectures, first one has to solve the approximate versions, then the sharp versions, and in case of the Loebl-Komlós-Sós conjecture we just have finished publishing the proof of the approximate version, while in case of the Erdős-Sós conjecture we have finished the proof of the Sharp version, and are writing it up.

We also solved problems in connection with graph-coloring (and choosability) and in connection with applications of combinatorics in coding theory (e.g., Füredi at al).

We have several results on Hamiltonian graphs, and on two-factors in graphs, further, on the length of the second longest cycle in Hamiltonian graphs.

Phase transition. Several results of ours reflect some sudden changes in the behaviour of the graphs when some graph parameters change a little around some critical values. These are sometimes called supersaturated structures, in other cases we speak of phase transitions. We have described several phase transition phenomena, among others a multiple phase transition in Ramsey-Turán theory (Balogh, Ping, Simonovits).

Regularity Lemma. Several of our results are connected to the applications of Szemerédi regularity lemma, and some of them (e.g. in case of Pósa-Seymour conjecture) on how to eliminate using the regularity lemma. A breakthrough was also achieved in connection with the famous El Zahar conjecture, (Szemerédi at al) where one has to cover a graph G_n on n vertices with k cycles of prescribed lengths.

Here we emphasize that there is nothing wrong with using this highly non-trivial tool (Regularity Lemma) to solve a problem, yet it is interesting to know when do we need it and when can we get rid of using it. Some of our results "attack" this questions, trying to solve some questions without regularity lemma, where earlier we used it. (See e.g., Szemerédi) One advantage of this may be that we get much-much better estimates on some parameters, or on the number of vertices from which on our results are applicable. (Using Regularity Lemma leads to results that hold only for very large parameters.)

Another direction is when we have an elementary but very complicated proof of some result and we try to make it more transparent, more understandable by using the regularity lemma. One of our results (Füredi) in this direction is when we prove the Erdős-Simonovits general stability theorem by using the Szemerédi Regularity Lemma and a very special case of the original theorem (which can be used very elementary, and effectively, by using symmetrization).

Extremal hypergraph problems. Füredi and Ruszinkó proved some deep results on Turán type extremal problems for very sparse excluded hypergraphs, this is connected to the famous Ruzsa-Szemerédi theorem. We proved several results connected to hypergraph generalizations of the Erdős-Gallai theorem, (Győri at al, Füredi at al) on hypergraph tree-extremal theorems and hypergraph cycle-extremal theorems, and other generalizations of the Erdős-Sós conjecture. (Surprisingly enough, in some cases – depending on the definitions – these results are easier than the corresponding graph results.) Also, we have results in connection with 2-factors in hypergraphs.

Some of our hypergraph extremal results are on linear hypergraph cycles (i.e., where any two hyperedge has at most one common vertex). In case of hypergraphs we may consider several different notions of cycles, e.g., loose cycles, strong cycles, **Berge cycles**, and **linear cycles**,... Actually we have several results in this area. One of the most surprising – and deepest - result is about 3-uniform hypergraphs not containing $(2k+1)$-cycles: surprisingly, the same order upper bound was proved (Győri at al) what we have for $2k$-cycles in graphs. Generalizations of this theorem for r-uniform or non-uniform hypergraphs were proved as well.

Another new research direction in extremal hypergraph theory, initiated by us: Győri, Gyárfás, and Simonovits, is when we have a 3-uniform hypergraph with a new kind of degree-condition. The **strong degree** of any vertex is the maximum number of triples containing a vertex x and vertex-disjoint outside x. Assuming that each vertex of a hypergraph has strong degree at least 3 implies the existence of a linear cycle. This new question surprised us by being very elementary and yet requiring a rather involved proof. Győri and his students later proved several theorems connected to this. In turn this led to some further interesting questions and results. (Here we restricted ourselves to the simplest non-trivial case: that is the case of 3-uniform hypergraphs.)

Here we also have to mention that some "intersection theorems" can be considered as hypergraph extremal results (e.g. Füredi and Frankl). Some of the results achieved by us belong to these category. Here we mention only

1. a new proof of the Erdős-Ko-Rado theorem,
2. Results connected to the Vapnik-Cervonenkis dimension, playing central role in some algorithmic questions (Füredi-Sali).

Data Structures. Some of the hypergraph results are strongly connected to codes and to data structures. Despite that our results mostly are of theoretical value, some of them are motivated by applications, e.g. by datastructures (Sali, Füredi).

Graph limits. Several directions in the theory of graph limits were investigated. This area has two well separated parts: the dense and the sparse case (where the sparse case is actually not only sparse, but refers to graphs of bounded degrees). We proved some new types of results on the dense case, (Lovász, Sós, and Vesztergombi were among those starting this research area). We investigated the relation between different characteristics, different parameters of large graphs, among others,

(1) about the "weak limit", (the statistics of densities of subgraphs with given number of vertices and edges, instead of considering the densities of every graph);

(2) on quasi-random properties, based on Janson's method, as the application of graph-limit theory, (Janson, Vera Sós). In our erlier papers we investigated the connection between quasi-randomness and regular partitions of graph sequences; here some questions are settled by using the theory of graph limits, combined with some methods developed by Janson.

(3) on graph-coloring, the statistics of the densities of monochromatic copies of fixed graphs.

(4) Connections to Statistical Physics (Borgs, Chayes, Lovász, Vera Sós, Kati Vesztergombi)...

Also we proved some results on the sparse case. We continued writing a book on the sparse case[2]

[2]the first larger book – by Lovász, more than 600pp – on graph limits came out not so

Some of our other results are related to limits of sequences of trees, more precisely, the metric limit theory of sequences of finite trees.

It is surprising that in these areas several classical, central results from measure theory can and should be used: measure theory has a deeper connection to classical combinatorics than what one would think. (See e.g. results of Gábor Elek and Balázs Szegedy, or of Miklós Abért and others.)

Other results. An earlier application (in combinatorial number theory) led Győri to the following theorem: any C_6-free graph can be made into a C_4-free graph deleting at most roughly half of the edges. Internationally several trials were attempted to prove the corresponding generalizations to C_{2k}-free graphs. (This would be that in any C_{2k}-free graph G one can find a subgraph G' with $e(G') > c_k e(G)$, for some constant $c_k > 0$.) Győri, Kensell and Tompkins proved the following nice result: if G does not contain any 6-cycle then we can keep 3/8 of the edges so that the resulting graph is bipartite and does not contain any 4-cycles. The conjecture was 2/5, but just recently Grosz, Methuku and Tompkins proved that actually 3/8 is the best possible.

We have several results on groups. Several results were proven on the distribution of roots of graph polynomials. These results were further developed by several other researchers. We also proved new types of results on applications of algebraic methods in combinatorial geometry (on estimating the number of high multiplicity numbers in case of several families of curves Elekes, Simonovits and Endre Szabó.)

Also some results are related with combinatorial number theory, some others with combinatorial geometry.

Impact. Perhaps one of the most important achievements is that a new school may emerge from our work, here, in Hungary, and internationally. Speaking of these new schools, we mean the extremal graph theoretical and the graph limit theoretical part. It was partly our group (Miklós Abért, Gábor Elek, and Balázs Szegedy) that started using very advanced methods to solve sparse graph limit questions.

In 2011 we organized a Paul Turán memorial conference, (not only in combinatorics, however, combinatorics was one of the most important part of it), 2013 was the 100th year anniversary of Paul Erdős' birth. These influenced our work in several ways.

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long ago., but of course, Lovász is not a member of our group.

(a) Several survey papers were written on these occasions, on the Mathematics of Paul Turán, (e.g., Simonovits), and of Paul Erdős (Füredi and Simonovits)...

(b) Several very important conferences were organized on these occasions. Our group had an important role in these conferences. One of the most important conferences was organized in Budapest, where our group participated very actively and successfully. (Another, perhaps the last one in this line was a Hungarian-Israeli Combinatorics conference, in Haifa, again, very successful.)

This actually meant that several members of our group were among the main organizers of these conferences, and they also gave lectures in the combinatorics section. (In case of the Erdős conference, we tried not to choose main organizers as plenary lecturers.) Also, there were several other Erdős conferences where we were invited to give main (plenary) lectures. And – since the participants of this group are internationally very acknowledged, we were also very often invited speakers on other combinatorial conferences, not necessarily connected with Erdős, often as plenary speakers.

Several of us were invited as colloquium speakers to various universities. Here I would emphasize the following things:

(a) Beside that the Old Hungarian School in Extremal Graph and Hypergraph theory is internationally very highly acknowledged, two new schools connected to our project emerged in the area, Abért-Elek-Szegedy and the Füredi-Győri "school".

(b) We could achieve important results, among others, in extremal hypergraph theory, which is a very difficult area.

(c) Our international impact, e.g., in China and India is becoming stronger and stronger. (It was always strong in the British combinatorial school.)